Non-standard quantum algebras and finite dimensional PT-symmetric systems.

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Outline

1. Construction of the $\mathcal{PT}$-symmetric $U_z(sl(2,\mathbb{R}))$ non-standard Hopft-algebra, finite dimensional bosonic representation.
2. Some families of $\mathcal{PT}$-symmetric Hamiltonians
3. Physical Application: to Quantum Dots.
4. Conclusions and Perspectives

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Non-Hermitian Quantum Mechanics

- **Operators:** $H \neq H^\dagger$
- **Eigenvalues:** complex.
- **Biorthogonality:** between eigenstates of H and $H^\dagger$

Pseudo-Hermitian QM:

$$\Upsilon H \Upsilon^{-1} = H^\dagger$$

(Particular case: PT-Symmetry Hamiltonians.)

$\boxed{\text{real or complex pair conjugate spectrum.}}$

$\mathcal{PT}$-Symmetric Hamiltonians.

- Parity Transformation ($\mathcal{P}$): $\hat{\mathcal{P}}\psi(x) = \psi(-x)$.
- Time Reversal ($\mathcal{T}$): $\hat{\mathcal{T}}\psi(t) = \psi^*(-t)$.
- H is $\mathcal{PT}$-symmetric if $[\mathcal{PT}, H] = 0$.

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Regions in the model-parameter space

- **Exact PT-symmetry phase:** The spectrum is real and its eigenvectors are also eigenvectors of the PT-symmetry operator.
  Symmetry Mapping:
  Exist Υ definite positive such that $h = \Upsilon^{1/2} H \Upsilon^{-1/2}$, $h = h^\dagger$.
- **Broken PT-symmetry phase:** The spectrum includes complex pair-conjugate eigenvalues, the eigenstates of the Hamiltonian are not eigenstates of the PT-symmetry operator.
- **Exceptional Points (EPs):** The boundary between these two regions. Two or more eigenvalues are degenerated and their eigenstates are coalescent.

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Non-standard quantum algebra $U_z(sl(2,\mathbb{R}))$.

J. Phys. A: Math. Gen. **29** (1996) L311–L316. Printed in the UK

LETTER TO THE EDITOR

Universal R-matrix for non-standard quantum $sl(\mathbf{2},\mathbb{R})$

Angel Ballesteros and Francisco J Herranz
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Received 26 March 1996

Abstract. A universal R-matrix for the non-standard (Jordanian) quantum deformation of $sl(2,\mathbb{R})$ is presented. A family of solutions of the quantum Yang–Baxter equation is obtained from some finite-dimensional representations of this Lie bialgebra quantization of $sl(2,\mathbb{R})$.

J. Phys. A: Math. Gen. **30** (1997) 6797–6809. Printed in the UK PII: S0305-4470(97)79997-9

Boson representations, non-standard quantum algebras and contractions

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Abstract. A Gelfan'd–Dyson mapping is used to generate a one-boson realization for the non-standard quantum deformation of $sl(2,\mathbb{R})$ which directly provides its infinite- and finite-dimensional irreducible representations. Tensor product decompositions are worked out for some examples. Relations between contraction methods and boson realizations are also explored in several contexts. So, a class of two-boson representations for the non-standard deformation of $sl(2,\mathbb{R})$ is introduced and contracted to the non-standard quantum (1+1) Poincaré representations. Likewise, a quantum extended Hopf $sl(2,\mathbb{R})$ algebra is constructed and non-standard quantum oscillator algebra representations are obtained from it by means of another contraction procedure.

$sl(2,\mathbb{R})$: $\{L_+, L_-, L_0\}$

$$[L_+, L_-] = 2L_0, \quad [L_0, L_\pm] = \pm L_\pm, \quad C = \frac{1}{2}L_0^2 + L_+L_- + L_-L_+ .$$

$sl(2,\mathbb{R})$
Deformation $z = 0$ $\leftarrow$ $U_z(sl(2,\mathbb{R}))$
Deformation $z \in \mathbb{R}$

$U_z(sl(2,\mathbb{R}))$: $\{j_+^{(z)}, j_-^{(z)}, j_0^{(z)}\}$

$$[j_0^{(z)}, j_+^{(z)}] = \frac{e^{2zj_+^{(z)}} - 1}{z}, \qquad \Delta(j_0^{(z)}) = 1 \otimes j_0^{(z)} + j_0^{(z)} \otimes e^{2zj_+^{(z)}}, \qquad \gamma(j_0^{(z)}) = -j_0^{(z)} e^{-2zj_+^{(z)}},$$

$$[j_0^{(z)}, j_-^{(z)}] = -2j_-^{(z)} + z{j_0^{(z)}}^2, \qquad \Delta(j_-^{(z)}) = 1 \otimes j_-^{(z)} + j_-^{(z)} \otimes e^{2zj_+^{(z)}}, \qquad \gamma(j_-^{(z)}) = -j_-^{(z)} e^{-2zj_+^{(z)}},$$

$$[j_+^{(z)}, j_-^{(z)}] = j_0^{(z)}, \qquad \Delta(j_+^{(z)}) = 1 \otimes j_+^{(z)} + j_+^{(z)} \otimes 1, \qquad \gamma(j_+^{(z)}) = -j_+^{(z)},$$

$$\epsilon(X) = 0,\ X \in \{j_0^{(z)}, j_+^{(z)}, j_-^{(z)}\}$$

$$C = \frac{1}{2} j_0 e^{-2zj_+^{(z)}} j_0 + \frac{1 - e^{-2zj_+^{(z)}}}{2z} j_- + j_- \frac{1 - e^{-2zj_+^{(z)}}}{2z} + e^{-2zj_+^{(z)}} - 1.$$

$sl(2,\mathbb{R})$:Finite dimensional bosonic realization, $[b, b^\dagger] = 1$.

$$\begin{aligned} L_+ &= b^\dagger, \\ L_- &= -(b^\dagger b + \beta) b, \\ L_0 &= 2b^\dagger b + \beta. \end{aligned}$$

$U_z(sl(2,\mathbb{R}))$: Finite dimensional bosonic realization, $[b, b^\dagger] = 1$.

$$\begin{aligned} j_+^{(z)} &= b^\dagger, \\ j_-^{(z)} &= -\frac{e^{2zb^\dagger}-1}{2z} b^2 - \beta \frac{e^{2zb^\dagger}+1}{2} b - z\beta^2 \frac{e^{2zb^\dagger}-1}{8}, \\ j_0^{(z)} &= \frac{e^{2zb^\dagger}-1}{z} b + \beta \frac{e^{2zb^\dagger}+1}{2}. \end{aligned}$$

Ballesteros, A, and Herranz, F. "Universal R-matrix for non-standard quantum."Journal of Physics A: Mathematical and General 29.13 (1996): L311.

Ballesteros, A, Herranz, F, and Negro, J. "Boson representations, non-standard quantum algebras and contractions."Journal of Physics

A: Mathematical and General 30.19 (1997): 6797.

The action of the operators $\{j_0^{(z)}, j_+^{(z)}, j_-^{(z)}\}$ on the eigenstates $\{|m\rangle, (m = 0, 1, \ldots \infty)\}$ of the usual boson number operator $b^\dagger b$, provides their lower-bounded representation, namely

$$
\begin{aligned}
j_+^{(z)}|m\rangle &= \sqrt{m+1}|m+1\rangle\,, \\
j_0^{(z)}|m\rangle &= (2m+\beta)|m\rangle + \sum_{k\geq 1}\frac{(2z)^k}{k!}\sqrt{\frac{(m+k)!}{m!}}\left(\frac{2m}{k+1}+\frac{\beta}{2}\right)|m+k\rangle\,, \\
j_-^{(z)}|m\rangle &= -\sqrt{m}(m-1+\beta)|m-1\rangle \\
&\quad -\sum_{k\geq 1}\frac{(2z)^k}{k!}\sqrt{\frac{(m+k)!}{m!}}\left[\frac{m}{\sqrt{m+k}}\left(\frac{m-1}{k+1}+\frac{\beta}{2}\right)|m-1+k\rangle + z\frac{\beta^2}{8}|m+k\rangle\right].
\end{aligned}
$$

$$C = \beta(\beta/2-1)$$

For values of the parameter $\beta \in \mathbb{Z}^-$, this representation becomes reducible and leads to the finite-dimensional irreducible representations of dimension $d = |\beta - 1|$ of the quantum algebra $U_z(sl(2,\mathbb{R}))$.

$sl(2,\mathbb{R})$: $\mathcal{PT}$-symmetric Finite dimensional bosonic realization.

$$L_+ = -\mathbf{i}b^\dagger, \quad L_- = -\mathbf{i}(b^\dagger b + \beta)b, \quad L_0 = 2b^\dagger b + \beta.$$

$\mathcal{PT} - sl(2,\mathbb{R})$
Deformation $z = 0$

$\leftarrow$

$\mathcal{PT} - U_z(sl(2,\mathbb{R}))$
Deformation $z \in \mathbb{R}$

$\mathcal{PT} - U_z(sl(2,\mathbb{R}))$

$$J_\pm^{(z)} = \mp\mathbf{i}j_\pm^{(-\mathbf{i}z)}, \quad J_0^{(z)} = j_0^{(-\mathbf{i}z)}.$$

$$J_+^{(z)} = -\mathbf{i}b^\dagger,$$

$$J_-^{(z)} = -\frac{e^{-2\mathbf{i}zb^\dagger}-1}{2z}b^2 - \mathbf{i}\beta\frac{e^{-2\mathbf{i}zb^\dagger}+1}{2}b - z\beta^2\frac{e^{-2\mathbf{i}zb^\dagger}-1}{8},$$

$$J_0^{(z)} = \mathbf{i}\frac{e^{-2\mathbf{i}zb^\dagger}-1}{z}b + \beta\frac{e^{-2\mathbf{i}zb^\dagger}+1}{2}.$$

A. Ballesteros, R. Ramírez, R, and MR, Journal of Physics A: Mathematical and General 57 (2024)035202.

Hamiltonian Construction

- Construction of two families of Hamiltonians with $\mathcal{PT}$-symmetric generators of $U_z(sl(2,\mathbb{R}))$ that are exactly solved.
- Analysis of spectra using similarity transformations and isospectral Hamiltonians.
- Study of parameter regions showing $\mathcal{PT}$-symmetry and broken $\mathcal{PT}$-symmetry.
- Identification of Exceptional Points (EPs) in the parameter space.

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Example 1:

$$\boxed{H(\mu_+,\mu_-,\mu_0) = \mu_- \, J_-^{(z)} + \mu_+[J_0^{(z)}, J_+^{(z)}] + \mu_0 J_0^{(z)}}$$

Given the operator : $\Upsilon = \mathrm{e}^{\eta\, J_0^{(z)}}\, \mathrm{e}^{\kappa\, J_+^{(z)}}\,, \quad \kappa = \pm\frac{1}{\mu_-}\left(\sqrt{\mu_0^2 + 2\mu_+\mu_-} - \mu_0\right).$

we have : $\boxed{h = \Upsilon H \Upsilon^{-1} = \mu_-\,\mathrm{e}^{-2\eta}\, J_-^{(z)} + z\mu_-\,\mathrm{e}^{-\eta}\,\sinh(\eta)\, J_0^{(z)2} \pm \sqrt{\mu_0^2 + 2\mu_+\mu_-}\; J_0^{(z)}}$

In the limit $\eta \to \infty$ reads

$$\boxed{h = \tfrac{1}{2} z\, \mu_-\, J_0^{(z)2} \pm \sqrt{\mu_0^2 + 2\mu_+\mu_-}\; J_0^{(z)}.}$$

Each irreducible representation of h, is represented by a triangular matrix of dimension $d = |\beta - 1|$ and the spectrum is

$$\sigma(h) = \begin{cases} \frac{(2k+1)^2}{2}\mu_- z \pm (2k+1)\sqrt{\mu_0^2 + 2\mu_+\mu_-}, & \text{even } d,\, 0 \le k \le \frac{d-2}{2} \\ \frac{(2k)^2}{2}\mu_- z \pm 2k\sqrt{\mu_0^2 + 2\mu_+\mu_-}, & \text{odd } d,\, 0 \le k \le \frac{d-1}{2}. \end{cases}$$

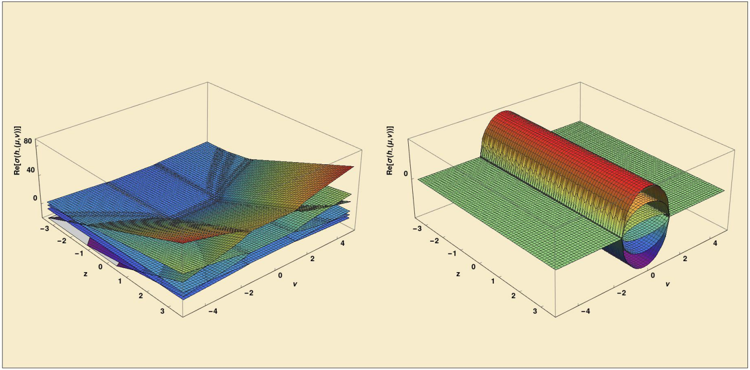


Figura: The spectrum of the Hamiltonian $h(\mu, \nu)$ in units of μ, as a function of ν and z, for dimensions $d = 6$, is depicted in Figure 3. In Panels (a) and (b) we plot the real and the imaginary part of the eigenvalues, respectively.

$$|\mu_-| = -|\mu_+| = \mu, \ \nu = \mu_0/\mu : \quad h(\mu,\nu)/\mu = \frac{1}{2} z \, J_0^{(z)2} + \sqrt{\nu^2 - 2} \, J_0^{(z)}.$$

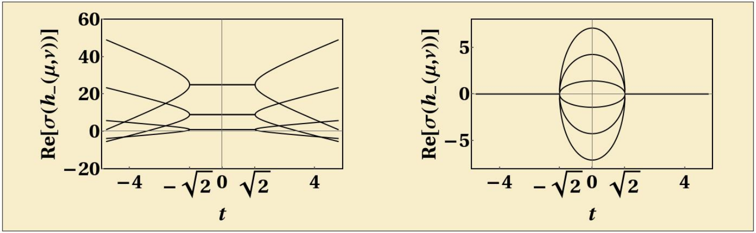


Figura: The spectrum of the Hamiltonian $h_-(\mu,\nu)$ in units of μ, as a function of ν and z, for dimensions $d = 6$, is depicted in Figure 3. In Panels (c) and (d), we present a cut in the graph for $z = 2$. The real and imaginary parts of the eigenvalues are presented in (c) and (d), respectively.

Example 2:

$$\boxed{H_0 = \mu_- J_-^{(z)} + \sum_{n=1}^{N} a_n \left[J_0^{(z)}\right]^n, \quad N \in \mathbb{Z}^+ \cup \infty}$$

$$\boxed{h_0 = \Upsilon H_0 \Upsilon^{-1} = \frac{z}{2}\mu_- \left[J_0^{(z)}\right]^2 + \sum_{n=1}^{N} a_n \left[J_0^{(z)}\right]^n}, \qquad \Upsilon = e^{\alpha\, J_0^{(z)}}.$$

$$\sigma(h_0) = \begin{cases} \frac{z}{2}\mu_-(2k-1)^2 + u_k^{\pm}, & \text{for } 1 \leq k \leq \frac{d}{2} \text{ and even } d \\ \frac{z}{2}\mu_-(2k-2)^2 + v_k^{\pm}, & \text{for } 1 \leq k \leq \frac{d+1}{2} \text{ and odd } d \end{cases} \tag{1}$$

where

$$\begin{aligned} u_k^{\pm} &= \sum_{n=1}^{N} (\pm 1)^n (2k-1)^n a_n, \\ v_k^{\pm} &= \sum_{n=1}^{N} (\pm 1)^n (2k-2)^{n+1} a_n. \end{aligned}$$

$$S(\mu_-,\lambda) = \mu_- J_-^{(z)} + \sin(\lambda J_0^{(z)}) \qquad C(\mu_-,\lambda) = \mu_- J_-^{(z)} + \cos(\lambda J_0^{(z)})$$

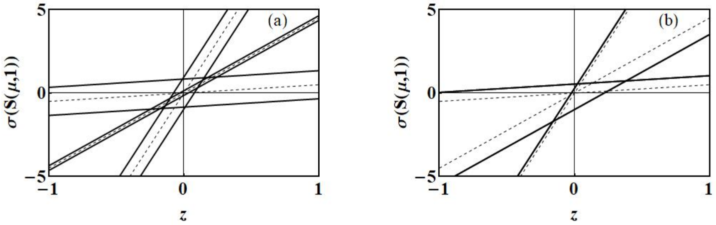


Figura: Spectra of the Hamiltonians $S(\mu_-,\lambda)$ and $C(\mu_-,\lambda)$ in units of μ_-, as a function of z, for dimension $d = 6$ and $\lambda = 1$ (solid lines). Dashed lines plot the spectrum of the Hamiltonian of h_0 for $a_n = 0\ \forall n$.

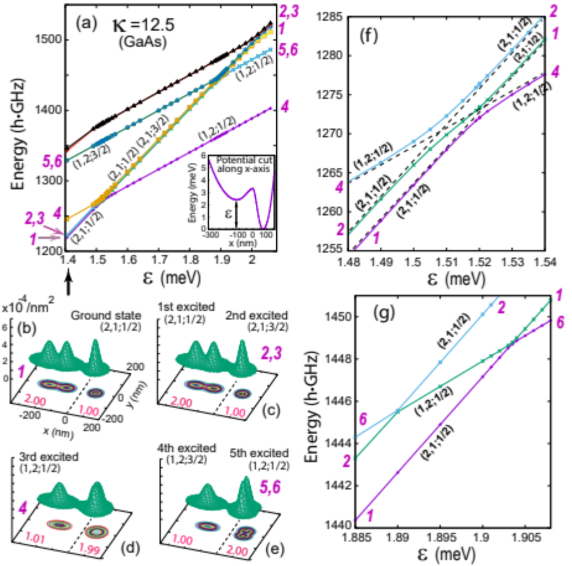
$$\begin{aligned} \sigma(S(\mu_-,1)) &= \frac{z}{2} \pm \sin(1),\ \frac{9}{2}z \pm \sin(3),\ \frac{25}{2}z \pm \sin(5), \\ \sigma(C(\mu_-,1)) &= \frac{z}{2} + \cos(1),\ \frac{9}{2}z + \cos(3),\ \frac{25}{2}z + \cos(5). \end{aligned}$$

Application to Quantum Dots

Yannouleas, Constantine, and Uzi Landman. "Wigner molecules and hybrid qubits."J. of Physics: Condensed Matter 34.21 (2022).

Three-electron hybrid qubits based on asymmetric GaAs double quantum dots-energy spectra as a function of the detuning parameter (FCI).

• GaAs (Gallium arsenide) semiconductor
devices such as integrated circuits at microwave frequencies.
• FCI: computational method to calculate the energy considering all possible configurations of electron interactions.
• Detuning (ε): the difference in energy levels between the two quantum dots in a double quantum dot system
control the distribution and interaction of electrons between the dots.


(a)
κ =12.5
(GaAs)
Energy (h·GHz)
ε (meV)
Potential cut along x-axis
Energy (meV)
x (nm)
(2,1;1/2)
(1,2;3/2)
(2,1;3/2)
(1,2;1/2)
(f)
(g)
x10^-4/nm^2
(b)
Ground state (2,1;1/2)
1st excited (2,1;1/2)
2nd excited (2,1;3/2)
(c)
3rd excited (1,2;1/2)
(d)
4th excited (1,2;3/2)
5th excited (1,2;1/2)
(e)
x (nm)
y (nm)
2.00
1.00
1.01
1.99

Consider the effective Hamiltonian

$$H_e = \begin{pmatrix} \delta L + \frac{\varepsilon}{2} & -t_3 & 0 & t_4 \\ -t_3 & -\frac{\varepsilon}{2} & t_1 & 0 \\ 0 & t_1 & \frac{\varepsilon}{2} & -t_2 \\ t_4 & 0 & -t_2 & \delta R - \frac{\varepsilon}{2} \end{pmatrix},$$

Coupling constants: $\delta L = 3$, $\delta R = 95.8$,
$t_1 = 1.8$, $t_2 = 7.1$, $t_3 = 11.5$, $t_4 = 6.3$ in units of $[GHz]$,
ε detunning parameter.
Eigenvalues (for ε sufficiently large):

$$\begin{aligned} E_{1,\pm} &= \frac{1}{2}\left(\delta L \pm \sqrt{(\delta L + \varepsilon)^2 + 4t_3^2}\right), \\ E_{2,\pm} &= \frac{1}{2}\left(\delta R \pm \sqrt{(\delta R - \varepsilon)^2 + 4t_2^2}\right). \end{aligned}$$

Effective Hamiltonian: H_{eff} of dimension $d = 4$ and with $z = \varepsilon$

$$H_{eff} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes H_1 + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes H_2,$$

with

$$\begin{aligned} H_1 &= \frac{1}{2}(\varepsilon + \delta L) J_0^{(\varepsilon)} + \frac{t_3^2}{\delta L} \varepsilon J_+^{(\varepsilon)} + \frac{\delta L}{\varepsilon} J_-^{(\varepsilon)}, \\ H_2 &= \frac{1}{2}(\varepsilon - \delta R) J_0^{(\varepsilon)} + \frac{t_2^2}{\delta R} \varepsilon J_+^{(\varepsilon)} + \frac{\delta R}{\varepsilon} J_-^{(\varepsilon)}. \end{aligned}$$

By a similarity transformations, S and P:

$$\begin{aligned}\mathfrak{h} &= PSH_{eff}S^{-1}P^{-1} \\ &= \begin{pmatrix} \delta L + \frac{\varepsilon}{2} & -t_3 & 0 & 0 \\ -t_3 & -\frac{\varepsilon}{2} & 0 & 0 \\ 0 & 0 & \frac{\varepsilon}{2} & -t_2 \\ 0 & 0 & -t_2 & \delta R - \frac{\varepsilon}{2} \end{pmatrix}.\end{aligned}$$

It is straightforward to prove that the eigenvalues of $\mathfrak{h}$ are

$$\begin{aligned}E_{1,\pm} &= \frac{1}{2}\left(\delta L \pm \sqrt{(\delta L + \varepsilon)^2 + 4t_3^2}\right), \\ E_{2,\pm} &= \frac{1}{2}\left(\delta R \pm \sqrt{(\delta R - \varepsilon)^2 + 4t_2^2}\right).\end{aligned}$$

$$S = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}, \qquad P = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix},$$

$$\begin{aligned}
s_1 &= \begin{pmatrix} \frac{\varepsilon}{2\delta L}\sqrt{-3\delta L^2 - 2\varepsilon\delta L + 4t_3^2} & 0 \\ 0 & 1 \end{pmatrix}, & p_1 &= \begin{pmatrix} \mathrm{i}\,\frac{\varepsilon}{2t_3}\sqrt{-3\delta L^2 - 2\varepsilon\delta L + 4t_3^2} & -\frac{1}{2t_3}(3\delta L + 2\varepsilon) \\ 0 & 1 \end{pmatrix}, \\
s_2 &= \begin{pmatrix} \frac{\varepsilon}{2\delta R}\sqrt{\delta R^2 - 2\varepsilon\delta R + 4t_2^2} & 0 \\ 0 & 1 \end{pmatrix}, & p_2 &= \begin{pmatrix} \mathrm{i}\,\frac{\varepsilon}{2t_2}\sqrt{\delta R^2 - 2\varepsilon\delta R + 4t_2^2} & \frac{1}{2t_2}(\delta R - 2\varepsilon) \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

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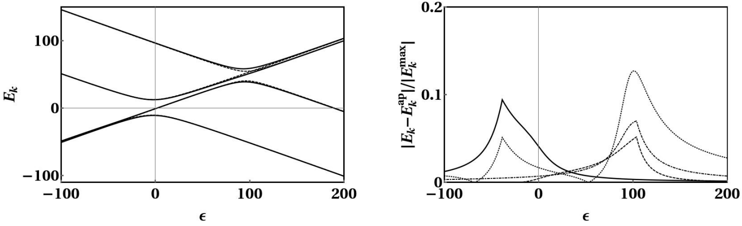


Figura: The Figure depicts the spectrum of H_e and H_{eff} as a function of ε. In Panel (a), the exact eigenvalues of H_e and the approximate values are displayed as a function of ε with solid and dashed lines, respectively. In Panel (b), we plot the absolute value of the difference between the exact and the approximate eigenvalue in units of the maximum or minimum absolute value of the exact solution at the point where each band avoids the crossing.

Work in progress.

Study of infinite dimensional representation and new families of Hamiltonians

$$
\begin{aligned}
J_0 &= \frac{e^{2zx}-1}{z}\partial_x - \lambda\frac{e^{2zx}+1}{2}, \\
J_+ &= x, \\
J_- &= -\frac{e^{2zx}-1}{2z}\partial_x^2 + \lambda\frac{e^{2zx}+1}{2}\partial_x - z\lambda^2\frac{e^{2zx}-1}{8}
\end{aligned}
$$

The $z \to 0$ limit is the usual second order differential realization of $sl(2,\mathbb{R})$

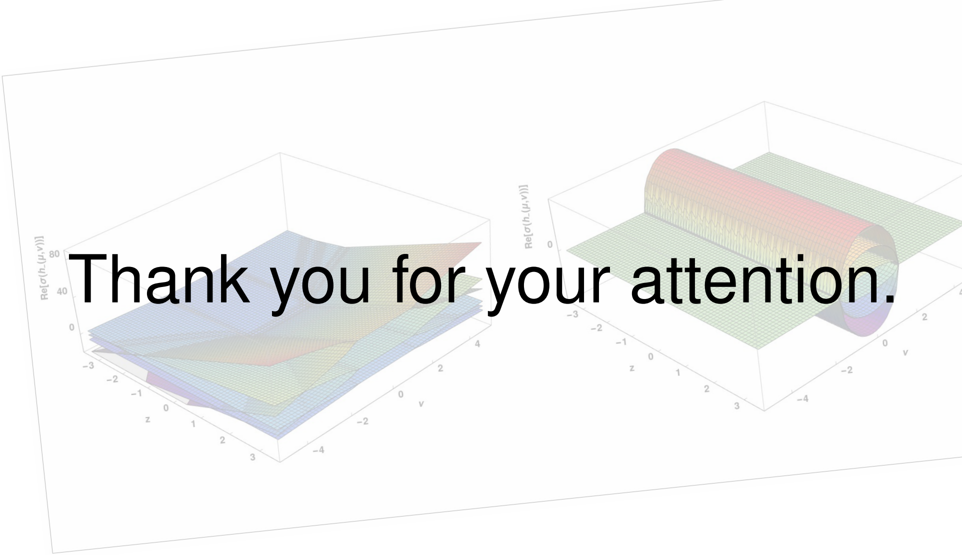
$$
L_0 = 2x\partial_x - \lambda, \qquad L_+ = x, \qquad L_- = -x\partial_x^2 + \lambda\partial_x,
$$

$$
H = f_+(J_+^{(z)})J_-^{(z)} + f_0(J_+^{(z)})J_0^{(z)} + f_+(J_+^{(z)}),
$$

where $f_0(J_+^{(z)}), f_\pm(J_+^{(z)})$ are functions of $J_+^{(z)}$.

Conclusions

- We have presented a boson realization of $U_z(sl(2,R))$ generators invariant under PT-transformation.
- We have derived the analytical spectrum for a family of PT-symmetric Hamiltonians using $U_z(sl(2,\mathbb{R}))$ quantum algebra.
- We have identified two families of PT-symmetric Hamiltonians with different spectral properties:
  - $H(\mu_+,\mu_-,\mu_0) = \mu_- J_-^{(z)} + \mu_+ [J_0^{(z)}, J_+^{(z)}] + \mu_0 J_0^{(z)}$:
    real spectrum: $\text{sign}(\mu_+) = \text{sign}(\mu_-)$
    complex conjugate pairs: $\text{sign}(\mu_+) = -\text{sign}(\mu_-)$.
  - $H_\mu = \mu_- J_-^{(z)} + J_+^{(z)}$
    real eigenvalues and band structures.
- Developed a non-standard quantum algebra model for three electron hybrid qubits in GaAs asymmetric double quantum dots.
- Future work: infinite-dimensional representation (position-dependent mass Hamiltonian).



Thank you for your attention.

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