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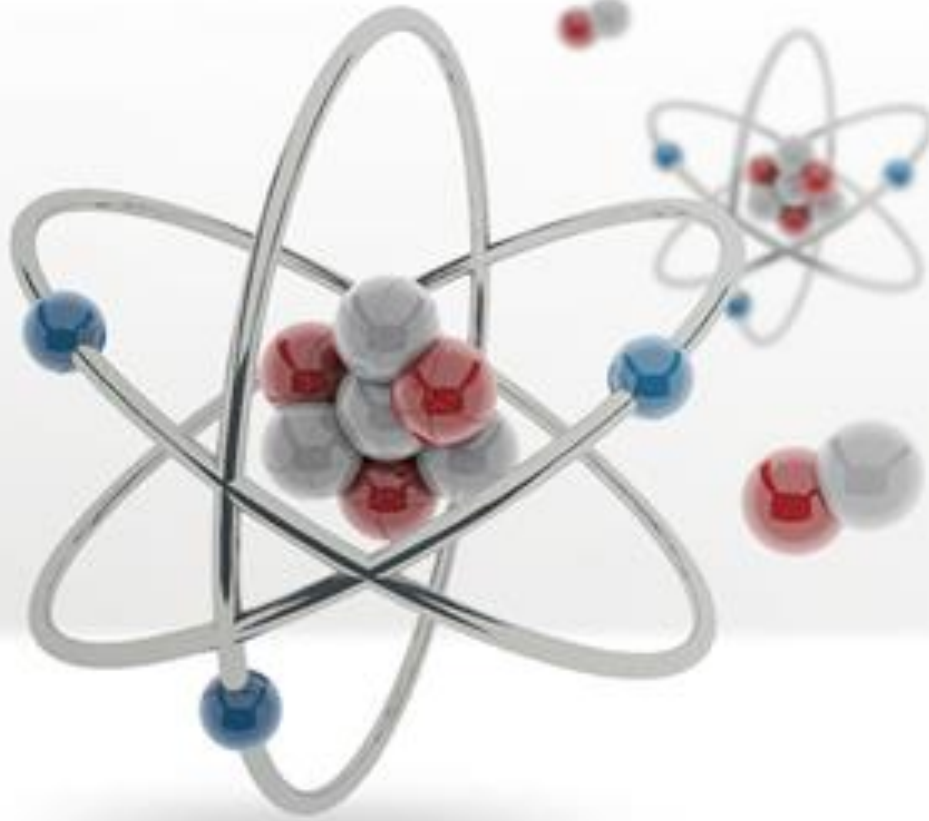


**Analitic and Algebraic Methods
in Physics (AAMP-2024)**

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Artur Ishkhanyan,
Viktor Red'kov**

A conditionally exactly solvable 1D Dirac pseudoscalar interaction potential

**AAMP-2024
Prague, August 27-30, 2024**



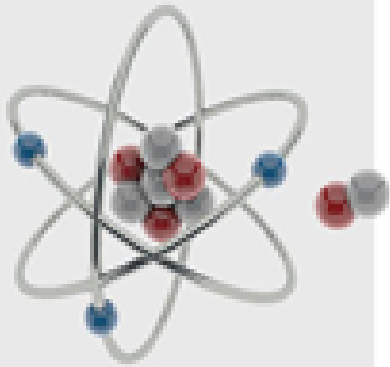
Outline

- ✓ 1D Dirac Equation
- ✓ Potential
- ✓ Solution and Bound States
- ✓ Energy Spectrum
- ✓ Discussion



*Paul Adrien
Maurice Dirac*
(1902-1984)

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1D stationary Dirac equation

$$E\sigma_0\psi = \left(-i\hbar\sigma_1 \frac{d}{dx} + W(x)\sigma_2 + mc^2\sigma_3 \right) \psi$$

$\sigma_{0,1,2,3}$ - *identity matrix, Pauli matrices*

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

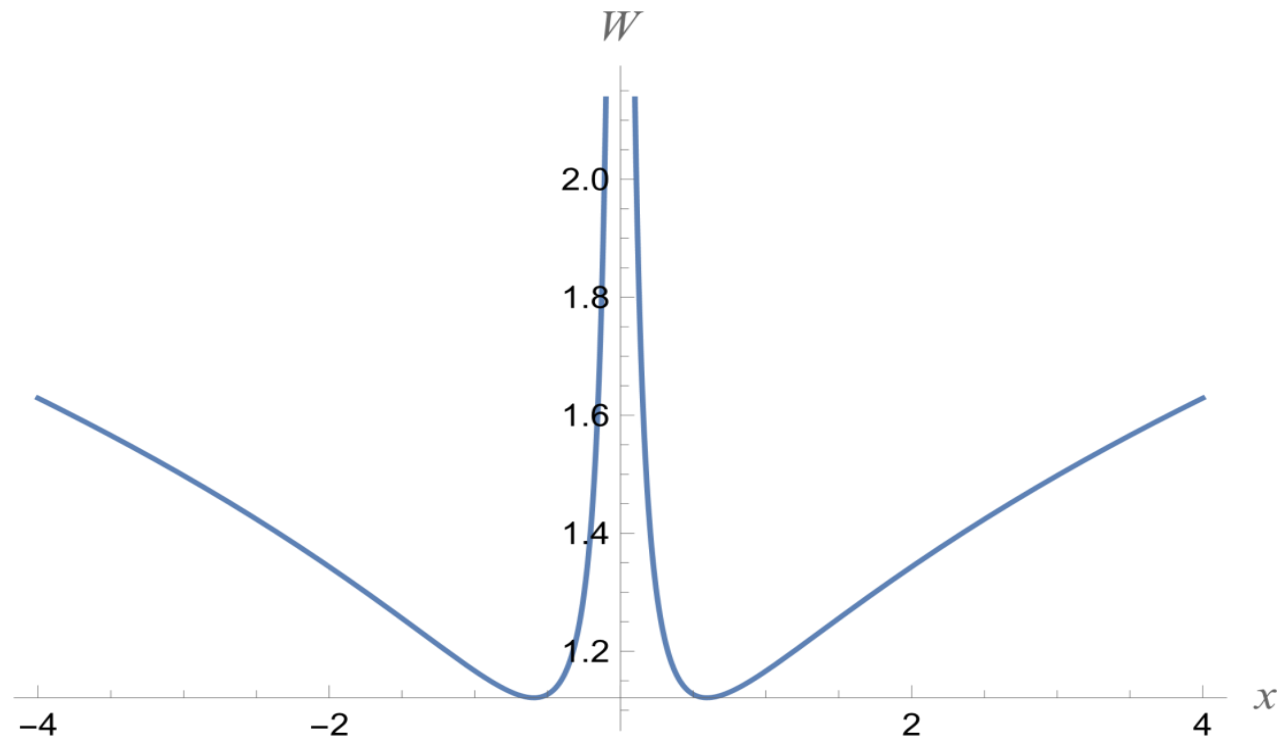
$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

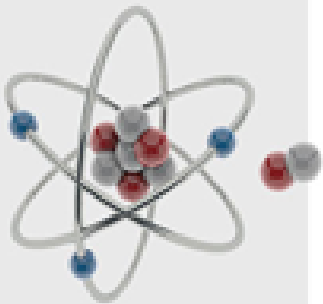
Dirac pseudoscalar potential

Basic Potential

$$W = \frac{c\hbar / 6}{|x|} + W_2 |x|^{1/3}$$



Symmetric with respect to the origin



Effective 1D stationary Schrödinger equation

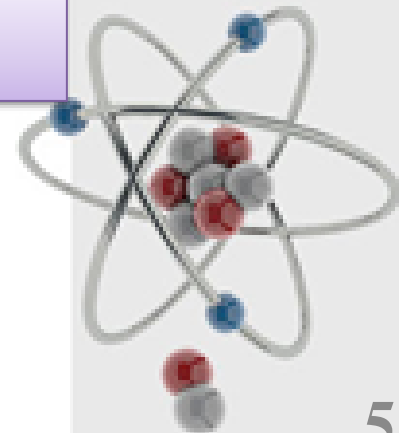
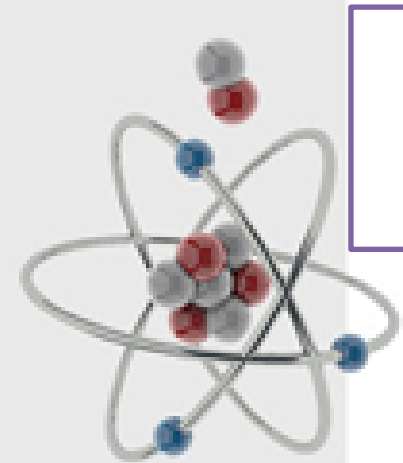
$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} (E_S - V_S(x))\psi = 0$$

Effective Energy

$$E_S = \frac{E^2 - m^2 c^4}{2mc^2}$$

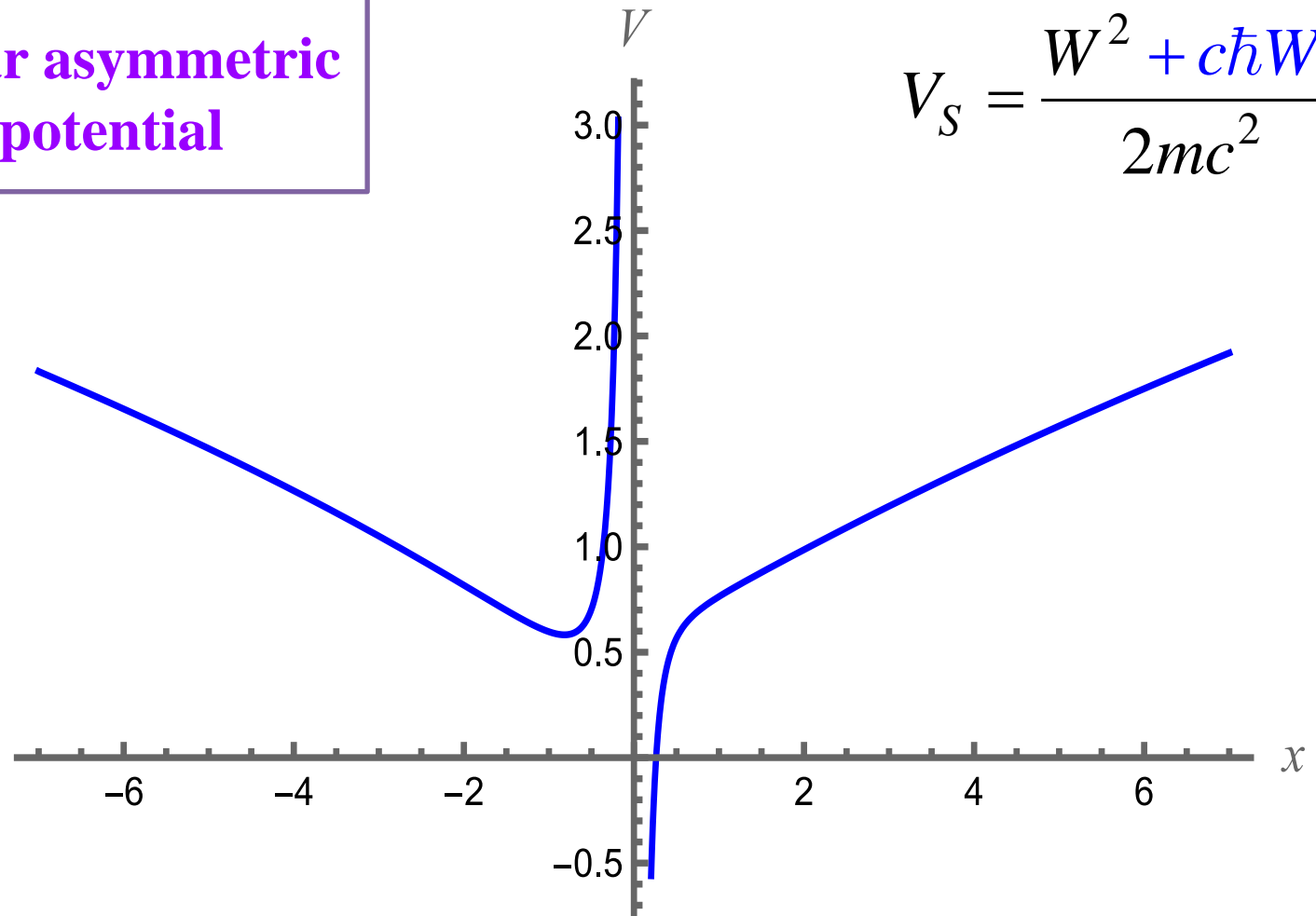
Effective potential

$$V_S = \frac{W^2 + c\hbar W'}{2mc^2}$$

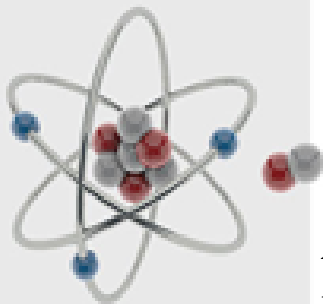


Effective Schrödinger potential

Weakly singular asymmetric
confining potential



$$V_S = \frac{W^2 + c\hbar W'}{2mc^2}$$



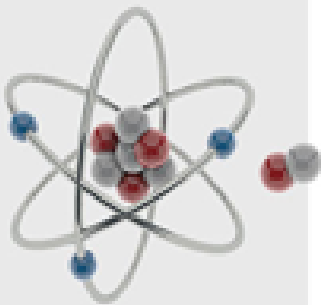
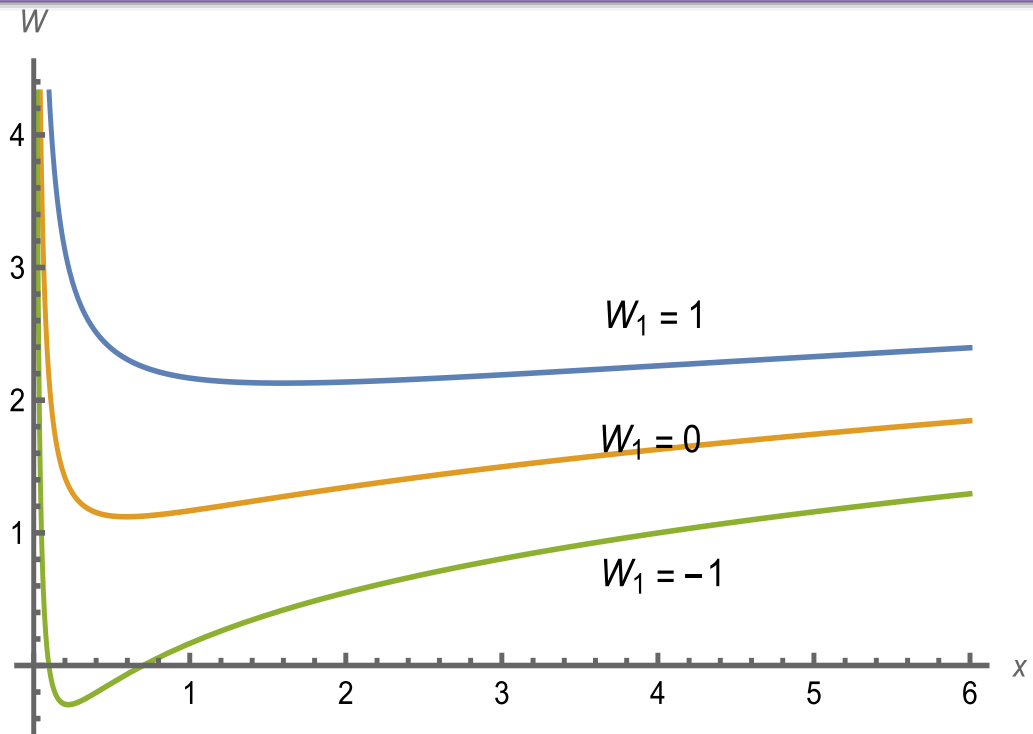
A.M. Ishkhanyan, V.P. Krainov, “Conditionally exactly solvable Dirac potential including $x^{1/3}$ pseudoscalar interaction”, *Phys. Scr.* **98**

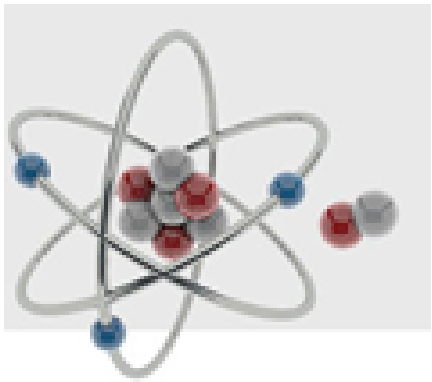
Dirac pseudoscalar interaction potential in the positive semi-axes $x > 0$

$$W = \frac{c\hbar / 6}{x} + \frac{W_1}{x^{1/3}} + W_2 x^{1/3}$$

For a positive W_2 this potential forms a potential well. The strength of this term defined by the value of the parameter W_1

$$W_2 = 1$$





$$-i\hbar \frac{d\psi_1}{dx} + iW\psi_1 = (E + mc^2)\psi_2$$

$$-i\hbar \frac{d\psi_2}{dx} - iW\psi_2 = (E - mc^2)\psi_1$$

Expressing ψ_2 from the first equation

$$\psi_2 = \frac{iW\psi_1 - i\hbar \psi_1'}{E + mc^2}$$

Substituting it into second equation, we obtain second-order equation for ψ_1

$$\frac{d^2\psi_1}{dx^2} + \frac{E^2 - m^2c^4 - c\hbar W' - W^2}{c^2\hbar^2} \psi_1 = 0$$

Effective Schrödinger-like equation

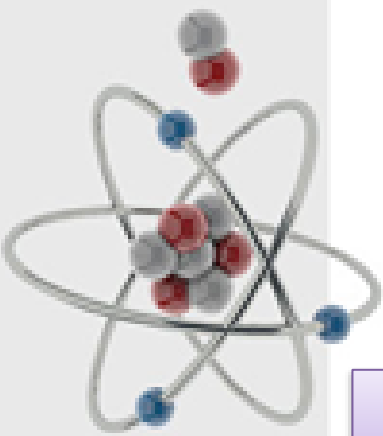
$$\frac{d^2\psi_1}{dx^2} + \frac{2m}{\hbar^2} (E_S - U_1(x)) \psi_1 = 0$$

Effective energy

$$E_S = \frac{E^2 - m^2c^4}{2mc^2}$$

Effective potential

$$U_1 = \frac{W^2 + c\hbar W'}{2mc^2}$$



$$-i\hbar \frac{d\psi_1}{dx} + iW\psi_1 = (E + mc^2)\psi_2$$

$$-i\hbar \frac{d\psi_2}{dx} - iW\psi_2 = (E - mc^2)\psi_1$$

Expressing ψ_1 from the second equation

$$\psi_1 = -\frac{iW\psi_2 + i\hbar\psi_2'}{E - mc^2}$$

Substituting it into first equation, we obtain second-order equation for ψ_2

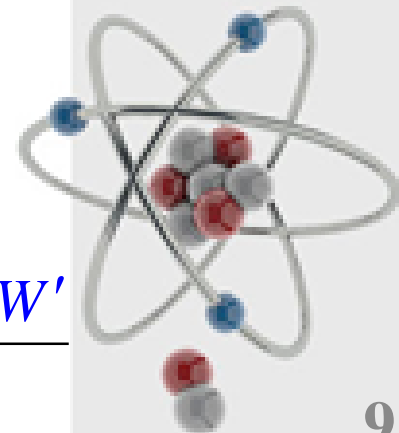
$$\frac{d^2\psi_2}{dx^2} + \frac{2m}{\hbar^2} (E_S - U_2(x))\psi_2 = 0$$

*Effective
energy*

$$E_S = \frac{E^2 - m^2c^4}{2mc^2}$$

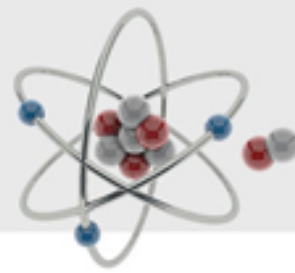
*Effective
potential*

$$U_2 = \frac{W^2 - c\hbar W'}{2mc^2}$$



1

Two effective Schrödinger potentials for the region $x > 0$



Effective Schrödinger Potential

$$U_1 = -\frac{5\hbar^2}{72mx^2} + \frac{W_1^2 + 2c\hbar W_2 / 3}{2mc^2 x^{2/3}} + \frac{W_1 W_2}{mc^2} + \frac{W_2^2}{2mc^2} x^{2/3}$$

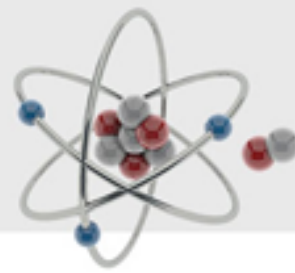
Particular case of *the first Stillinger potential*

$$V_{S1} = -\frac{5\hbar^2}{72mx^2} + \frac{V_1}{x^{2/3}} + V_0 + V_2 x^{2/3}$$

This potential is *attractive* in the vicinity of the origin

2

Two effective Schrödinger potentials for the region $x > 0$



Effective Schrödinger Potential

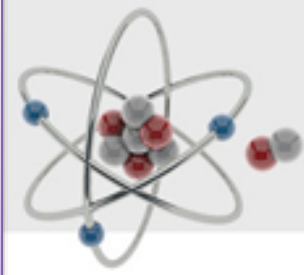
$$U_2 = \frac{7\hbar^2}{72mx^2} + \frac{\hbar W_1}{3mcx^{4/3}} + \frac{W_1^2}{2mc^2x^{2/3}} + \frac{W_1W_2}{mc^2} + \frac{W_2^2}{2mc^2}x^{2/3}$$

Particular case of *the second Exton potential*

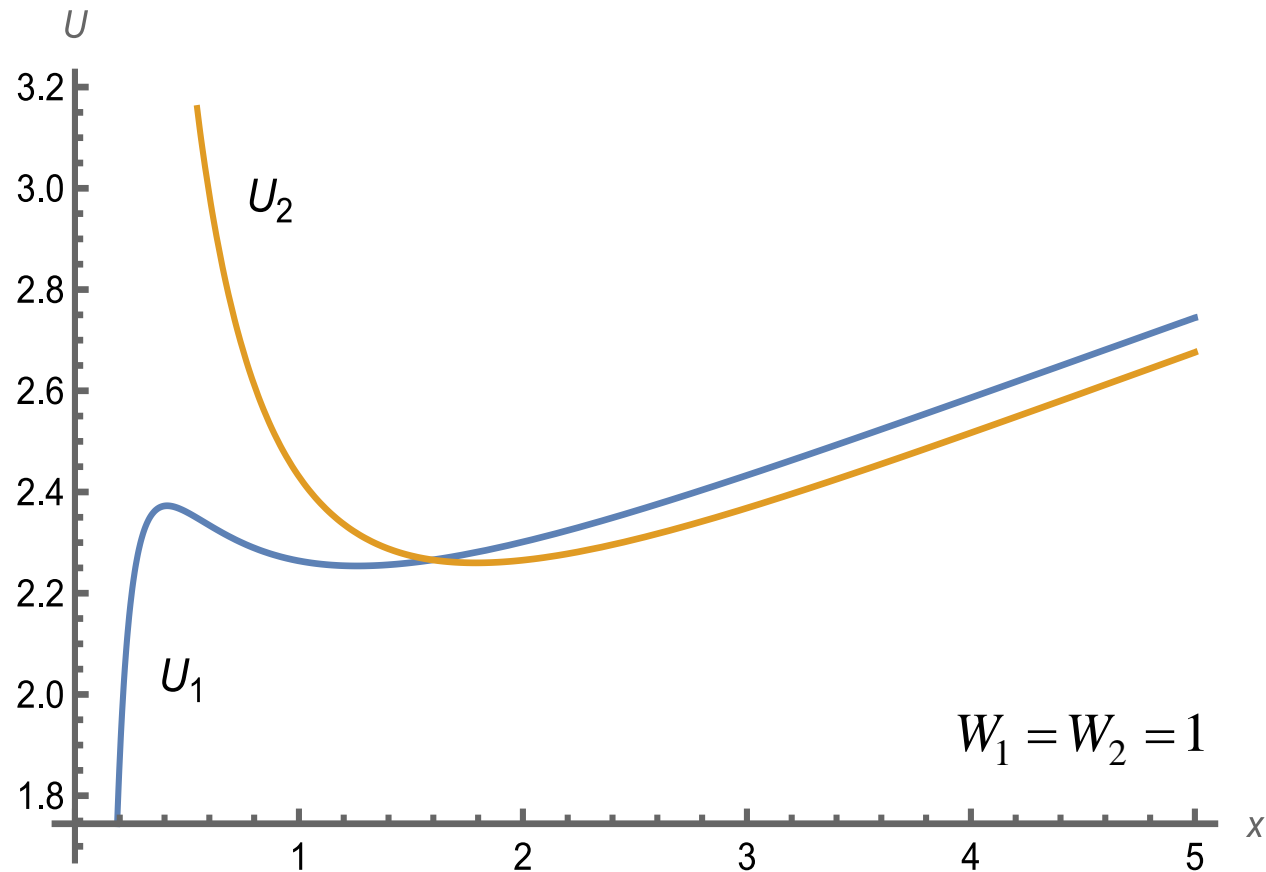
$$V_{E2} = \frac{7\hbar^2}{72mx^2} + \frac{V_1}{x^{4/3}} + \frac{9mV_1^2}{2\hbar^2x^{2/3}} + V_0 + V_2x^{2/3}$$

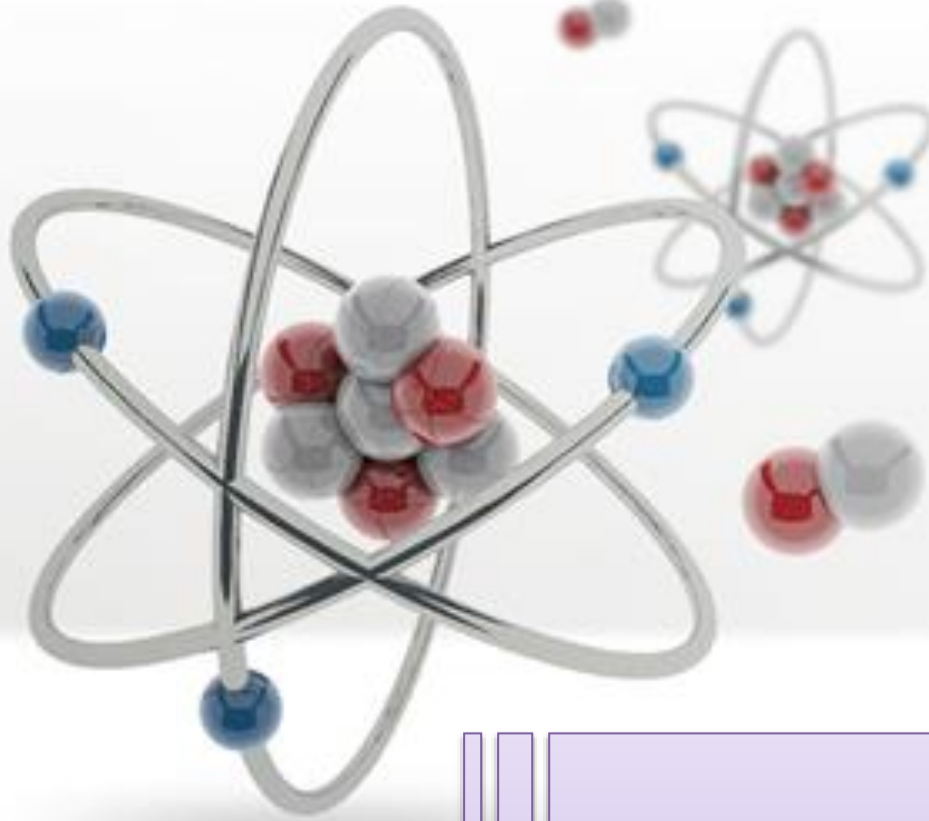
This potential is *repulsive* in the vicinity of the origin

Effective Schrödinger potentials

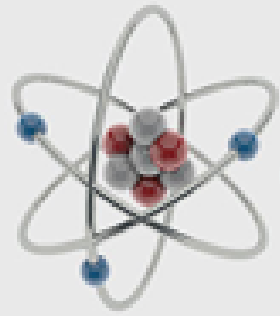


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Solution and bound states



Solution of the Schrödinger equation for the first Stillinger potential

$$\frac{d^2\psi_1}{dx^2} + \frac{2m}{\hbar^2} (E_S - U_1(x))\psi_1 = 0$$

$$\psi_1(x) = x^{1/6} e^{a-(z-A)^2/2} \left(C_1 H_{a-1}(z-A) + C_2 \cdot {}_1F_1\left(\frac{1-a}{2}, \frac{1}{2}, (z-A)^2\right) \right)$$

$$A = \sqrt{\frac{3(E^2 - 2W_1W_2 - m^2c^4)^2}{8c\hbar W_2^3}}$$

$$a = \frac{3(E^2 - m^2c^4)(E^2 - 4W_1W_2 - m^2c^4)}{16c\hbar W_2^3}$$

$$z = \sqrt{\frac{3W_2}{2c\hbar}} x^{2/3}$$

Wave function for bound states on the interval $x \in (0, +\infty)$

$$\psi_{1R} \Big|_{x \rightarrow +\infty} = 0 \Rightarrow C_2 = 0$$

$$\psi_1 = C_1 x^{1/6} e^{a-(z-A)^2/2} H_{a-1}(z-A)$$

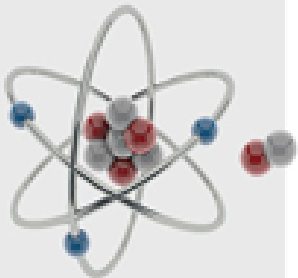
$x \rightarrow 0:$

$$\psi_1 = C_1 e^{a-A^2/2} \left(A_1 x^{1/6} + B_1 x^{5/6} + O\left(x^{3/2}\right) \right)$$

$$A_1 = H_{a-1}(-A)$$

$$B_1 = -\sqrt{\frac{3W_2}{2c\hbar}} \left(H_a(-A) + AH_{a-1}(-A) \right)$$

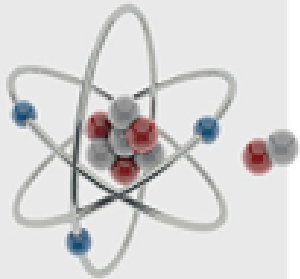
$$A = \sqrt{2a}$$



Two wave function components are not independent
They are connected via the following relations

$$\psi_1 = -\frac{iW\psi_2 + i\hbar\psi_2'}{E - mc^2} \qquad \psi_2 = \frac{iW\psi_1 - i\hbar\psi_1'}{E + mc^2}$$

The relation between wave function components does not allow
 $A_1 = 0$



It is impossible !!!
Why?

Wave component ψ_2

$$\psi_2 = i \frac{C_1 e^{a-A^2/2}}{E + mc^2} \left(\frac{A_2}{x^{1/6}} + B_2 x^{1/2} + O(x^{7/6}) \right)$$

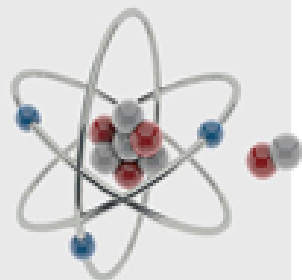
$$A_2 = -\frac{2c\hbar}{3} B_1 + W_1 A_1$$

$$B_2 = W_2 (2a - A^2) A_1 + W_1 B_1$$

$$A_2 \neq 0 \Rightarrow \psi_2(x)$$

has a singularity at the origin

A fundamental requirement of quantum mechanics is that the wave function is finite everywhere



A_2

must be zero

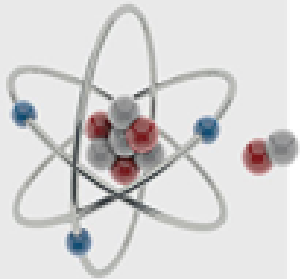
Exact eigenvalue equation for the bound state energy spectrum

$$A_2 = 0:$$

$$F = H_a(-A) + \left(A + \frac{\sqrt{3}W_1}{\sqrt{2c\hbar W_2}} \right) H_{a-1}(-A) = 0$$

Approximate energy spectrum when $W_1 = 0$

$$E_n \approx \pm \sqrt{m^2 c^4 + W_2^{3/2} \sqrt{\frac{16c\hbar}{3} \left(n - \frac{1}{6} \right)}}$$



Maslov index

$$\gamma_M = -\frac{1}{6}$$

Approximation of the eigenvalue equation

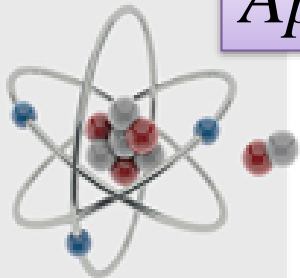
For **high-lying** levels, one can apply the Airy-function approximation to the Hermite function for the **left transition region** $-A \approx -\sqrt{2a}$

$$F \approx f(a) \sin(\pi(a - \gamma_M)) = 0$$

$$\gamma_M = -\frac{1}{\pi} \tan^{-1} \left(\frac{Ai(u_1) - {}^{12}\sqrt{\frac{2a+1}{2a-1}} \frac{A+s}{\sqrt{2a}} Ai(u_2)}{Bi(u_1) - {}^{12}\sqrt{\frac{2a+1}{2a-1}} \frac{A+s}{\sqrt{2a}} Bi(u_2)} \right) \quad u_{1,2} = 2^{1/3} (2a \pm 1)^{1/6} (A - \sqrt{2a \pm 1})$$

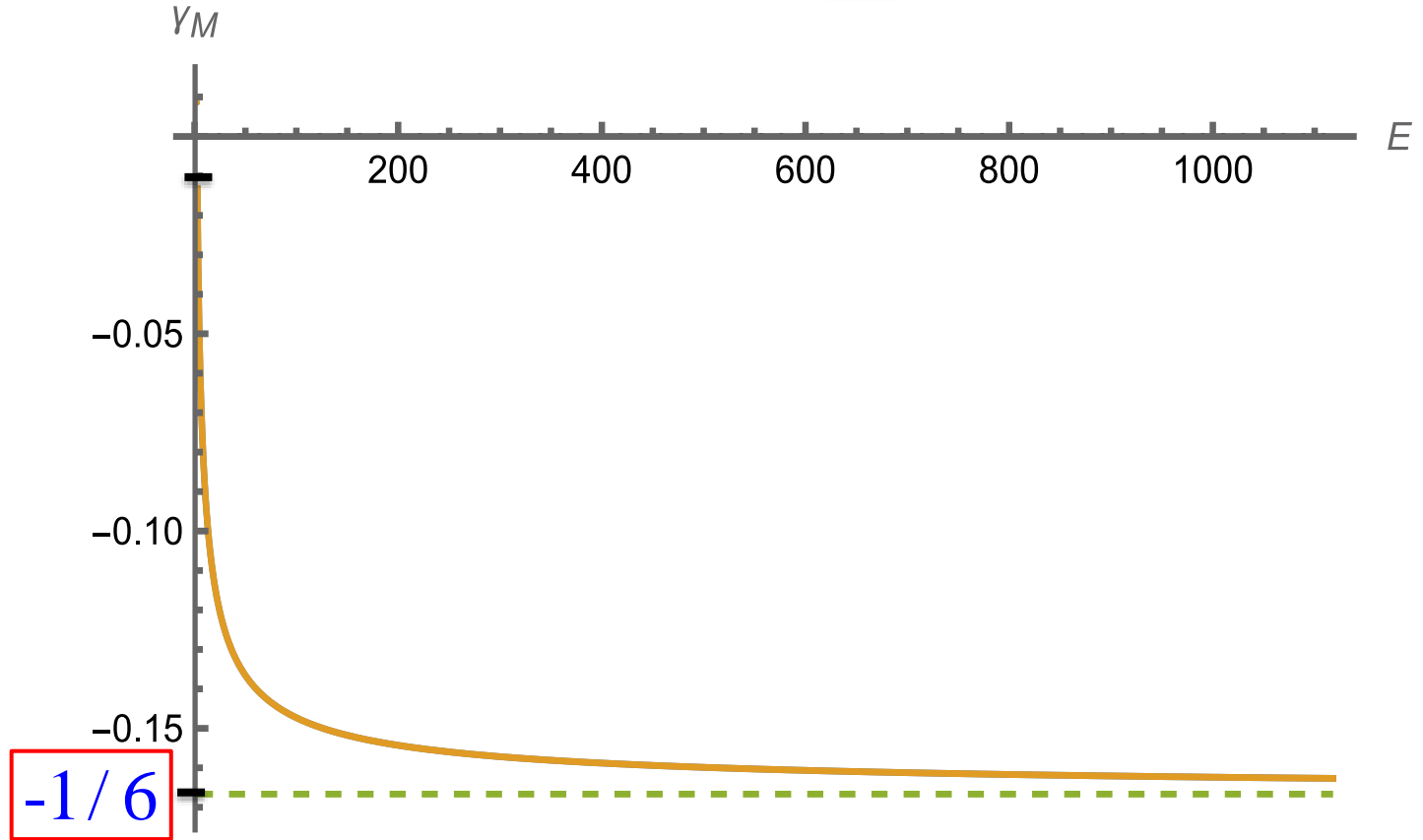
$$s = \frac{\sqrt{3}W_1}{\sqrt{2c\hbar W_2}}$$

Approximate spectrum with the potential term $W_1 x^{-1/3}$

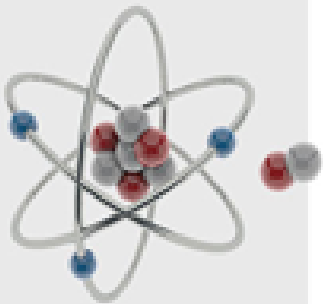


$$E_n \approx \pm \sqrt{m^2 c^4 + 2W_1 W_2 + 2W_2 \sqrt{W_1^2 + \frac{4c\hbar W_2}{3} \left(n - \frac{1}{6} \right)}}$$

The dependence of γ_M on energy

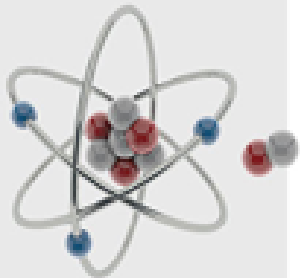


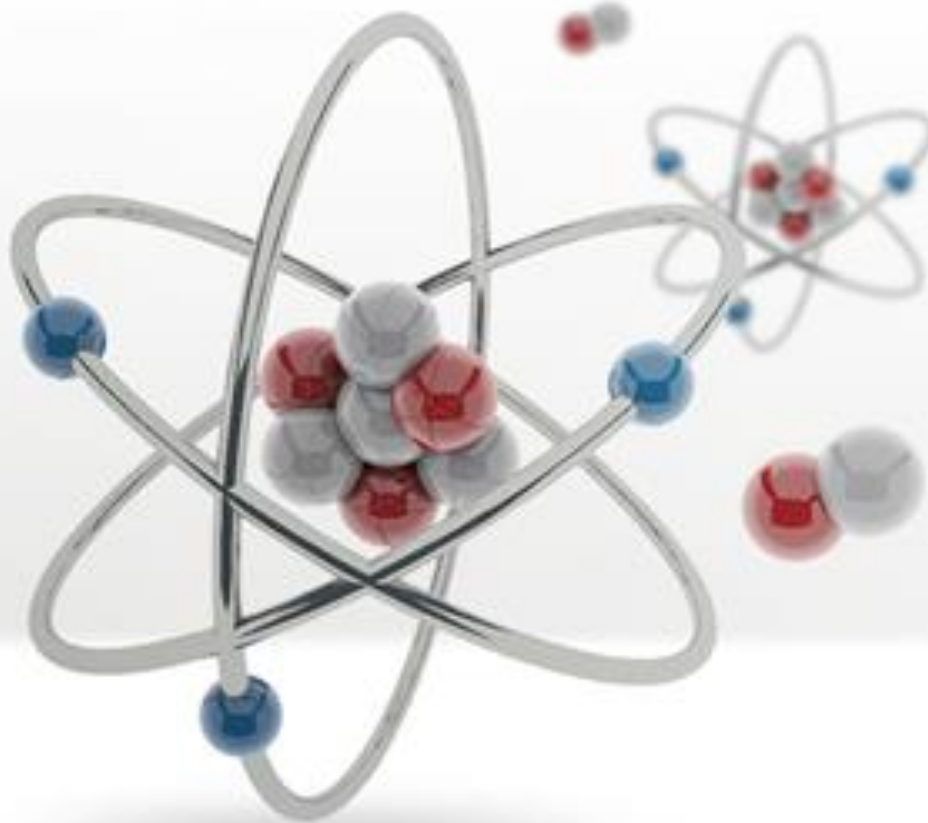
The black mark is not zero, it depends on the value of W_1, W_2



Discussion

- We have examined an analytically solvable pseudoscalar interaction potential for the one-dimensional stationary Dirac equation
- The general solution to the Dirac equation is written in terms of non-integer index Hermite functions and confluent hypergeometric functions
- We have derived the exact equation for the energy spectrum, developed an approximation for the spectrum
- Our results demonstrate that the inclusion of the $x^{-1/3}$ term has a significant impact on the energy spectrum and eigenfunctions, particularly, on the low-lying energy levels.
- These results indicate that the pseudoscalar interaction potential with an $x^{-1/3}$ term can be used to model a variety of physical systems, particularly low-dimensional systems such as graphene, semiconductor nanostructures, or topological insulators, where relativistic effects and confinement are significant factors





Future Directions

- ✓ Addition of Vector Potential

$$V(x) = \frac{V_0}{x^{2/3}} + V_1 x^{2/3}$$

- ✓ Scalar Potential

- ✓ The case $x < 0$

- ✓ ...

- ✓ Applications

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THANK YOU