Explicit n-particle harmonic oscillator states

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- The combination of two spin-$\frac{1}{2}$ systems gives:
one *antisymmetric* singlet $S = 0$ and
one *symmetric* triplet $S = 1$.
- The permutation symmetry uniquely identifies S when taking multiple copies of spin-1/2 states.
- In combining single-particle harmonic oscillator states of the same parity ($su(1,1) \sim sp(2,\mathbb{R})$ states), the permutation group is ***not enough***.

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- Construct *some* many-particle $sp(4,\mathbb{R})$ states “easily”
- have an idea of where the permutation symmetry is hiding.

su(1,1) oscillator states

- The $su(1,1) \sim sp(2,\mathbb{R})$ operators satisfy

$$[\mathsf{K}_0, \mathsf{K}_\pm] = \pm \mathsf{K}_\pm \,, \qquad [\mathsf{K}_+, \mathsf{K}_-] = -\mathsf{K}_0 \,.$$

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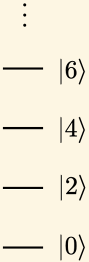

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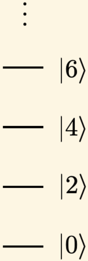

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- Natural for boson systems

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Where do bosons actually belong?

A Marzuoli[1,2], F A Raffa[3] and M Rasetti[3,4]

Combining two su(1,1) systems

- $K_- = \frac{1}{2}\left(\hat{a}_1\hat{a}_1 + \hat{a}_2\hat{a}_2\right)\,,\quad K_+ = K_-^\dagger\,,\quad K_0 = \frac{1}{2}\left(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2 + 1\right)$

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- There are infinitely many towers of symmetric states.

$\vdots$

—

— $|4\rangle|0\rangle + \sqrt{\frac{2}{3}}|2\rangle|2\rangle + |0\rangle|4\rangle$

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— $|0\rangle|0\rangle$

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$$\begin{array}{ll} \vdots & \\ \text{---} & \\ \text{---} & |4\rangle|0\rangle - |0\rangle|4\rangle \\ \text{---} & |2\rangle|0\rangle - |0\rangle|2\rangle \end{array}$$

- There are infinitely many towers of antisymmetric states.

Combining two su(1,1) systems

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- Clearly permutation symmetry is not enough to identify the tower of states.

The Laplacian approach

$$\begin{array}{lll}
\vdots & \vdots & \vdots \\
\text{---} & \text{---} & \text{---} \\
\text{---}\ |4\rangle|0\rangle + \sqrt{\tfrac{2}{3}}|2\rangle|2\rangle + |0\rangle|4\rangle & \text{---}\ |4\rangle|0\rangle - |0\rangle|4\rangle & \text{---}\ |4\rangle|0\rangle - \sqrt{6}|2\rangle|2\rangle + |0\rangle|4\rangle \\
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K_- |0\rangle|0\rangle &= 0\,, \\
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- Write

$$\hat{a}_i \mapsto \frac{\partial}{\partial x_i}\,, \qquad \hat{a}_j^\dagger \mapsto x_j \quad \Rightarrow \quad K_- \mapsto \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

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- The bottom states map to polynomials:

$$\begin{aligned} |0\rangle|0\rangle &\mapsto f_0 = 1\,, \\ |2\rangle|0\rangle - |0\rangle|2\rangle &\mapsto f_2 = x_1^2 - x_2^2\,, \\ |4\rangle|0\rangle - \sqrt{6}|2\rangle|2\rangle + |0\rangle|4\rangle &\mapsto f_4 = x_1^4 - 6x_1^2x_2^2 + x_2^4 \end{aligned}$$

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- The f_k are two-dimensional harmonic functions.

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- In the expansion of $H_2^+(x_1, x_2)$, distinct powers of h_2 give distinct functions f_k that satisfy $\nabla^2 f_k = 0$.
- Only even numbers of excitations are possible so even powers of x_1 or x_2 must be kept:

$$1 + \frac{1}{2} h_2^2 (x_1^2 - x_2^2) + \frac{1}{24} h_2^4 (x_1^4 - 6x_1^2 x_2^2 + x_2^4) + \ldots$$

The Laplacian approach

- For three particles:

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} = 0.$$

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- There's also a generating function for the solutions to this.

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Advances in Applied Clifford Algebras

Generating Functions for Spherical Harmonics and Spherical Monogenics

P. Cerejeiras, U. Kähler and R. Lávička*

Connection with spherical harmonics

- Again keeping only terms in even powers of x_i:

$$1 + \frac{1}{2}(x_1^2 - x_2^2) + \frac{1}{2}(x_1^2 + x_2^2 - 2x_3^2)$$

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are all at the bottom of separate 3-particle towers.

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- There is in fact a symmetry to the solutions: angular momentum plus parity.

Three particles and $O(3)$

- For 3 particles we have:

$$K_+ = \tfrac{1}{2}\left(\hat{a}_1^\dagger \hat{a}_1^\dagger + \hat{a}_2^\dagger \hat{a}_2^\dagger + \hat{a}_3^\dagger \hat{a}_3^\dagger\right), \qquad K_- = K_+^\dagger ,$$
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- Of course $[\hat{L}_j, \hat{L}_k] = i\epsilon_{jk\ell}\hat{L}_\ell$
- Also $[\hat{L}_j, K_q] = 0$.

Three particles and $O(3)$

- Thus, all states in a tower are also "good angular momentum states", *i.e.* they have good quantum numbers ℓ and $|m|$.

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- The states $|\ell, m; n)$ in $\ell = 0, m = 0$ tower:

$$\begin{aligned}
Y_0^0(\theta,\varphi) &= 1 & &\sim |0\rangle|0\rangle|0\rangle & &= |00;0) \\
r^2 Y_0^0(\theta,\varphi) &= x_1^2 + x_2^2 + x_3^2 & &\sim |2\rangle|0\rangle|0\rangle + |0\rangle|2\rangle|0\rangle + |0\rangle|0\rangle|2\rangle & &= |00;2) \\
r^4 Y_0^0(\theta,\varphi) &= (x_1^2 + x_2^2 + x_3^2)^2 & & & &= |00;4)
\end{aligned}$$

Three particles and $O(3)$

- The $\ell = 4, m = 0$ tower are built on:

$$r^4 Y_4^0(\theta, \varphi) \sim |40;4) = |4\rangle|0\rangle|0\rangle + \sqrt{\tfrac{2}{3}}|2\rangle|2\rangle|0\rangle - 4\sqrt{\tfrac{2}{3}}|2\rangle|0\rangle|2\rangle + \tfrac{8}{3}|0\rangle|0\rangle|4\rangle$$

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- We also have $\ell = 4, |m| = 2$ and $|m| = 4$ towers built on:

$$\begin{aligned} r^4\left(Y_4^2(\theta,\varphi) + Y_4^{-2}(\theta,\varphi)\right) &\sim |42;4) \sim |4\rangle|0\rangle|0\rangle - \sqrt{6}|2\rangle|0\rangle|2\rangle \\ &\qquad - |0\rangle|4\rangle|0\rangle + \sqrt{6}|0\rangle|2\rangle|2\rangle \\ r^4\left(Y_4^4(\theta,\varphi) + Y_4^{-4}(\theta,\varphi)\right) &\sim |44;4) \sim |4\rangle|0\rangle|0\rangle - \sqrt{6}|2\rangle|2\rangle|0\rangle + |0\rangle|4\rangle|0\rangle \end{aligned}$$

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$$\mathrm{P}_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{is an orthogonal matrix: } \mathrm{P}_{12}^{\top} = \mathrm{P}_{12}^{-1}$$

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Single-particle irrep of $sp(4, \mathbb{R})$

- We have the generators

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A_{ij} &= a_i^\dagger a_j^\dagger \,, && i = 1, 2 \,, \\
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- Because A_{ij}'s commute only the symmetric part of the tensor product is needed.

Irreps of $sp(4, \mathbb{R})$

- The symmetric part of the repeated coupling of $(2, 0)$ irreps contains states in the $\mathfrak{u}(2)$ irreps

$$D = (00) + (20) + (40) + (22) + (60) + (42) + (22) + \ldots$$

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- For single particles irrep of $sp(4,\mathbb{R})$ this bottom state is carries the $(0,0)$ irrep so D lists the $\mathfrak{u}(2)$ contents of the single particle irrep.

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Two-particle irreps of $sp(4, \mathbb{R})$

- We now have

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- The lowering operators

$$B_{11} \mapsto \frac{\partial^2}{\partial x_{11}^2} + \frac{\partial^2}{\partial x_{21}^2} ,$$
$$B_{12} \mapsto \frac{\partial^2}{\partial x_{11} \partial x_{12}} + \frac{\partial^2}{\partial x_{21} \partial x_{22}} ,$$
$$B_{22} \mapsto \frac{\partial^2}{\partial x_{12}^2} + \frac{\partial^2}{\partial x_{22}^2} .$$

Two-particle irreps of $\mathfrak{sp}(4, \mathbb{R})$

- Not so easy to find the functions f *simultaneously* solutions to

$$\left(\frac{\partial^2}{\partial x_{11}^2} + \frac{\partial^2}{\partial x_{21}^2}\right) f = 0\,,$$
$$\left(\frac{\partial^2}{\partial x_{11}\partial x_{12}} + \frac{\partial^2}{\partial x_{21}\partial x_{22}}\right) f = 0$$
$$\left(\frac{\partial^2}{\partial x_{12}^2} + \frac{\partial^2}{\partial x_{22}^2}\right) f = 0\,.$$

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- Use $O(2)$ by constructing the tensors

$$T^1 = \frac{1}{2}\left(a^\dagger_{11} - i a^\dagger_{21}\right), \quad V^1 = \frac{1}{2}\left(a^\dagger_{12} - i a^\dagger_{22}\right).$$

$$a^\dagger_{i\alpha}: \quad \text{particle number } i\text{, mode number } \alpha.$$

- Note that

Clarify what is L12

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- Example: $|m| = 4$ irrep has lowest state

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- Compare the $su(1,1)$ states at the bottom of $|m| = 4$:

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- Use $O(3)$ by constructing the tensors T^L_M and V^L_M:

$$
\begin{aligned}
T^1_{-1} &= -\frac{1}{2}\left(a^\dagger_{11} + i a^\dagger_{21}\right) \,, \quad T^1_0 = -\frac{1}{\sqrt{2}} a^\dagger_{31} \,, \quad T^1_1 = \frac{1}{2}\left(a^\dagger_{11} - i a^\dagger_{21}\right) \,, \\
V^1_{-1} &= -\frac{1}{2}\left(a^\dagger_{12} + i a^\dagger_{22}\right) \,, \quad V^1_0 = -\frac{1}{\sqrt{2}} a^\dagger_{32} \,, \quad V^1_1 = \frac{1}{2}\left(a^\dagger_{12} - i a^\dagger_{22}\right) \,.
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- Compare the $su(1,1)$ states at the bottom of $L=2, |m|=0$:

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$$\sum_{mm'} \left\langle \begin{smallmatrix} 2 \\ m \end{smallmatrix} ; \begin{smallmatrix} 2 \\ m' \end{smallmatrix} \,\middle|\, \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \right\rangle T_m^2 V_{m'}^2 |00\rangle|00\rangle|00\rangle$$

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- T^2 and V^2 are $L = 2$ tensors under $O(3)$, and the combo $\sum_{mm'} \left\langle {2 \atop m}; {2 \atop m'} \;\middle|\; {3 \atop 2} \right\rangle T_m^2 V_{m'}^2$ is an "axial tensor" in the sense that it lives in the antisymmetric space of the decomposition $(L = 2) \otimes (L = 2)$.

Three-particle irreps of $sp(4, \mathbb{R})$

- Some don't exist:

$$\sum_{mm'} \left\langle {2 \atop m}; {2 \atop m'} \;\middle|\; {1 \atop 0} \right\rangle T^2_m V^2_{m'} |00\rangle|00\rangle|00\rangle$$

  does not yield product states with only even number of excitations for each particles.
- This is because $m = 0$ irreps of $O(2)$ (as a subgroup of $O(3)$) must have even parity but the coupling is antisymmetric in T and V.

Conclusion

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- It is also helpful for 2-particle irreps of $sp(2k,\mathbb{R})$.
- Some extra care required for $n \geqslant 3$ particles

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