FLOW FROM GENERIC COMPLEX NON-HERMITIAN MATRICES TO NORMAL MATRICES AND THE FATE OF THE SINGLE RING THEOREM

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Work (in progress) in collaboration with Roman Riser (Texas Tech U Lubbock), Richard Scalettar (UC Davis) and Tony Zee (KITP-UC Santa Barbara)

The Single Ring Theorem

JF & Antony Zee, NPB ***501****(1997) 643-669*

For any matrix probability ensemble of the form $P(\phi,\phi^\dagger) = \frac{1}{\mathcal{Z}}\, e^{-N\mathrm{tr}V(\phi^\dagger\phi)}$, with $V(u)$ an admissible analytic function of its argument,

the shape of the support of the average density of eigenvalues $\rho(x,y) = \langle \frac{1}{N}\sum_{i=1}^{N}\delta(x-\mathrm{Re}z_i)\delta(y-\mathrm{Im}z_i)\rangle$

of the complex $N\times N$ matrix ϕ in the complex plane, in the limit $N\to\infty$, ***is either a disk or an annulus***, centered at the origin.

comments:

$\mathcal{Z}$ is a normalization factor

$V(\phi^\dagger\phi)$ can be a polynomial of arbitrarily high degree, hence possibly with many minima

the ensemble is rotational invariant: $\mathrm{Prob}(\phi) = \mathrm{Prob}(e^{i\alpha}\phi) \Longrightarrow \rho(x,y) = \frac{\rho(r)}{2\pi}$

where $\rho(r)$ is the radial density of eigenvalues ($r = |z| = |x+iy|$).

normalization $\int\limits_0^\infty \rho(r) r dr = 1$

proof given in

ELSEVIER Nuclear Physics B 501 [FS] (1997) 643-669

Non-gaussian non-hermitian random matrix theory: Phase transition and addition formalism

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by summing exactly all double-line ('t Hooftian) planar Feynman diagrams

sketch of the proof:

the Green's function $F(w) = \langle \frac{1}{N} \mathrm{tr} \frac{1}{w - \phi^\dagger \phi} \rangle$ (the Cauchy transform of the averaged spectral density of) the positive hermitian random matrix $\phi^\dagger \phi$ can be computed in the large-N limit by using standard methods of hermitian RMT (e.g., Dyson gas techniques).

asymptotic behavior $F(w) \underset{w \to \infty}{\simeq} \frac{1}{w}$ (normalization to unity of the density of eigenvalues of $\phi^\dagger \phi$)

radial counting function (integrated radial density of eigenvalues) $\gamma(r) = \int_0^r \rho(r') r' dr'$

$\gamma(r)$ is monotonically increasing from $\gamma(0) = 0$ to $\gamma(\infty) = 1$ and is constant on segments on which $\rho(r) = 0$

In NPB501(1997)643, it was proved that $F(w)$ and $\gamma(r)$ are related functionally by the equation $\gamma \left[r^2 F\left(\frac{\gamma r^2}{\gamma - 1} \right) - \gamma + 1 \right] = 0$

from which the SRT follows, without explicit solution of the latter equation, simply by seeking constant (r-independent) solutions for $\gamma(r)$

and using the asymptotic behavior $F(w) \underset{w \to \infty}{\simeq} \frac{1}{w}$

later additional analytical results and numerical demonstration in

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"Single ring theorem" and the disk-annulus phase transition

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a pedagogical review (based on my talk in PHHQP IV in Stellenbosch 2005)

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Non-Hermitian random matrix theory: summation of planar diagrams, the 'single-ring' theorem and the disc–annulus phase transition

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Qualitative argument in favor of the Single Ring Theorem (SRT)

Singular Value Decomposition (SVD):

$$\phi = U\Lambda V, \quad U, V \in U(N)$$

singular values: $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)\,, \quad \lambda_i \geq 0$

comments:

$\phi^\dagger\phi = V^\dagger\Lambda^2 V\,, \quad \phi\phi^\dagger = U\Lambda^2 U^\dagger$ **isospectral, positive**

U, V **not unique, determined up to a `gauge transformation'** $V \sim \Theta V\,, \quad U \sim U\Theta^\dagger\,, \quad \Theta \in U(1)^N$

the 2^N -fold sign ambiguity in Λ is fixed by absorbing the signs of individual λ_i into U or V

parameter counting: $2N^2$ **real parameters in ϕ =** $N^2 + N^2$ **real parameters in $U\&V$**
$+N$ **real parameters in Λ** $-N$ **gauged away real parameters (phases) in Θ**

from the SVD obtain $\phi = U(\Lambda W)U^\dagger\,,\quad W = VU$

hold W fixed as U wanders freely throughout $U(N)$

thus obtain a family of unitarily equivalent matrices, which are isospectral to the fiducial matrix $\phi_f = \Lambda W$

(a `jargon guy' would lavishly say ϕ is the orbit of ϕ_f in $U(N)$ under adjoint action.)

$\Longrightarrow$ **enough to study the spectrum of complex eigenvalues of ϕ_f for given Λ, W**

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the argument:

assume further that the spectrum of Λ splits into several disjoint segments, which could be realized in our ensemble when $V(\phi^\dagger\phi)$ has several well-separated minima, where the eigenvalues of Λ^2 condense.

in an unrestricted complex matrix model $U\&V$ are independent of each other, and therefore as they vary independently, W roams through the entire group $U(N)$ causing the complex eigenvalues of $\phi_f = \Lambda W$ to smear in the complex plane, and not be concentrated in annuli with radii given by the minima of $V(\Lambda^2)$

comment: this argument explains why the number of eigenvalue rings in the complex plane is bounded from above by the number of minima of $V(\Lambda^2)$

to see that at the number of rings cannot be larger than one requires the detailed proof, in the large-N limit.

This argument suggest a way to evade the SRT:
correlate $U\&V$ in a way which will prevent W roaming freely through the unitary group $U(N)$

for example, consider a block-diagonal W, with the upper diagonal block a $K \times K$ unitary diagonal matrix $\mathrm{diag}(e^{i\omega_1}, e^{i\omega_2}, \ldots, e^{i\omega_K})$ with K a finite fraction of N. Thus, if the first K eigenvalues of Λ^2 clump into several disjoint segments, the corresponding first K eigenvalues of ϕ will form rings (or annuli) in the complex plane whose radii follow those clumps.

In the extreme case $K = N$, we'll have a completely diagonal $W = \mathrm{diag}(e^{i\omega_1}, e^{i\omega_2}, \ldots, e^{i\omega_N})$, which would result in all eigenvalues of ϕ forming rings (or annuli) according to the breakup of the spectrum of Λ^2 into segments.

In this case $\phi_f = \Lambda W = \mathrm{diag}(\lambda_1 e^{i\omega_1}, \lambda_2 e^{i\omega_2}, \ldots, \lambda_N e^{i\omega_N})$, rendering $\phi = U\phi_f U^\dagger$ a normal matrix $[\phi, \phi^\dagger] = 0$

Conclusion:
Normal matrices (drawn from a rotationally invariant probability ensemble) evade the SRT <u>maximally</u>

This motivates us to consider a parametric family of Random Matrix Models of the form

$$P_c(\phi,\phi^\dagger) = \frac{1}{\mathcal{Z}_c}\, e^{-N\mathrm{tr}\left(V(\phi^\dagger\phi)+\frac{c}{4}[\phi,\phi^\dagger]^2\right)}$$

with positive parameter c .

$[\phi,\phi^\dagger]$ is hermitian, and therefore $[\phi,\phi^\dagger]^2$ is a positive matrix.

Consequently, $\mathrm{tr}[\phi,\phi^\dagger]^2 = 0 \Longleftrightarrow [\phi,\phi^\dagger] = 0$, that is, $\Longleftrightarrow \phi$ a normal matrix.

In summary, $c\,\mathrm{tr}[\phi,\phi^\dagger]^2$ penalizes ϕ in probability for deviation from normality.

This model flows (interpolates continuously) between generic (rotationally invariant) complex random matrix models and normal matrix models, with flow parameter c:

At $c=0$ the model coincides with our original ensemble $P(\phi,\phi^\dagger) = \frac{1}{\mathcal{Z}}\, e^{-N\mathrm{tr}V(\phi^\dagger\phi)}$

As c is turned on, the penalty term increasingly suppresses non-normal matrices,

until only normal matrices ϕ are allowed in the limit $c\to\infty$

Thus, as c is turned on and increased, we expect the effect of normal matrices at some point to be strong enough and evade the SRT.

Comment: note that $\mathrm{tr}[\phi,\phi^\dagger]^2 = 2\mathrm{tr}[(\phi^\dagger\phi)^2 - \phi^2\phi^{2\dagger}]$. Thus the first part of the penalty term term proportional to $\mathrm{tr}(\phi^\dagger\phi)^2$ can be combined with the original potential $\mathrm{tr}V(\phi^\dagger\phi)$, and their combined effect can be studied in the large-N limit by summing all planar diagrams as before. However, the other part $\mathrm{tr}(\phi^2\phi^{2\dagger})$ in the penalty term amounts to a new type of interaction vertex which breaks planarity as soon as c is turned on.

This complicates the diagrammatic analysis tremendously, as one would have to go beyond leading planar order in the large-N expansion, and resum non-perturbatively infinitely many non-planar diagrams.

Thus, at this point we resort to numerical investigations…

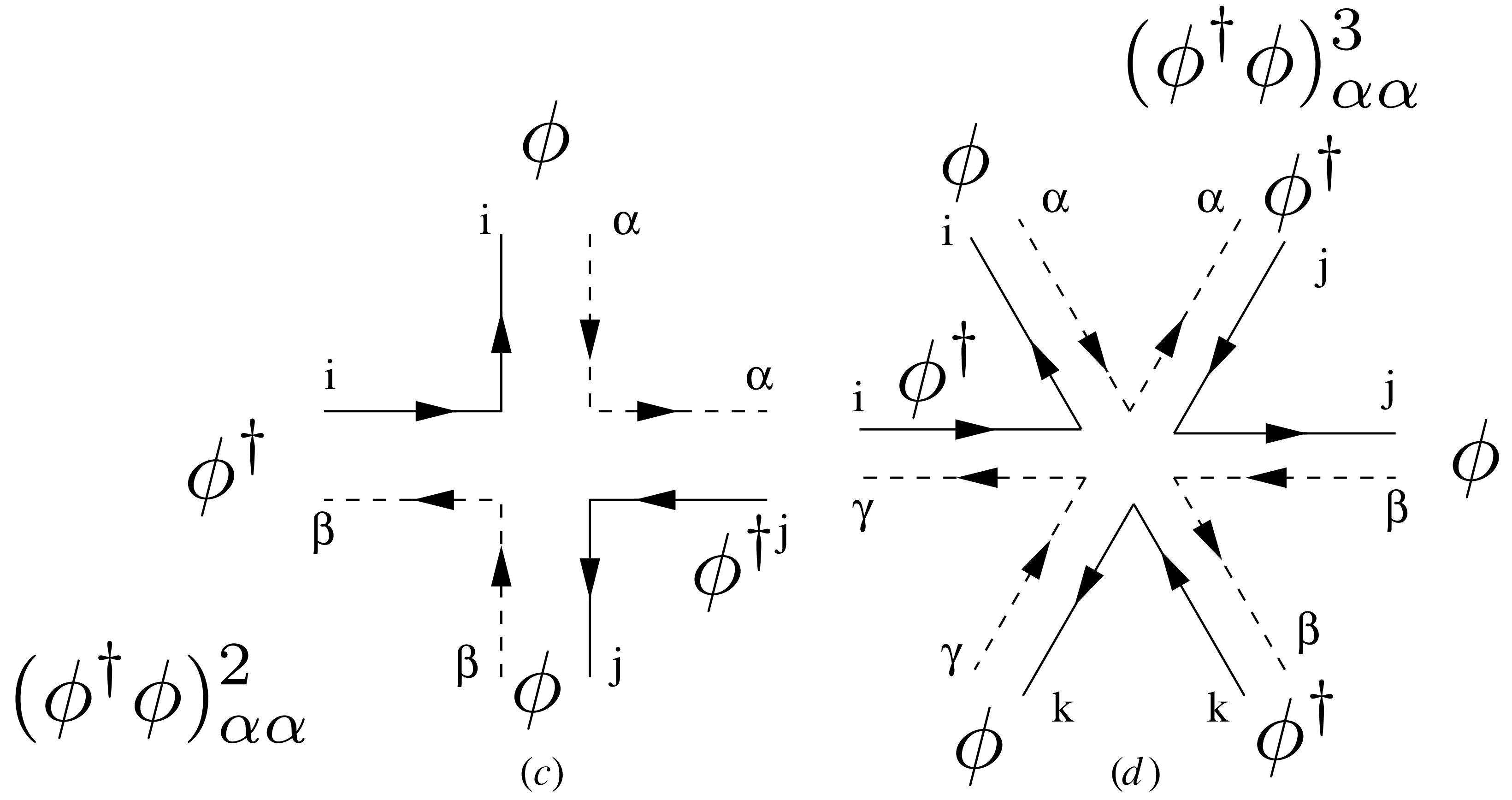


Figure 2. The two bare quark–gluon vertices (*a*) and (*b*) and quartic (*c*) and sextic (*d*) gluon self-interaction vertices. All vertices are of order *N*.

Monte-Carlo simulations of the flow model

Study concretely the case of cubic V: $V(\phi^\dagger\phi) = m^2\phi^\dagger\phi + \frac{g}{2}(\phi^\dagger\phi)^2 + \frac{u}{3}(\phi^\dagger\phi)^3$

which is the simplest nontrivial polynomial potential in which the eigenvalues of $\phi^\dagger\phi$ (i.e., Λ^2) could coalesce in two disjoint segments.

Indeed, for $g < 0$ it has two local maxima on the positive real axis

and the eigenvalues of Λ^2 will coalesce in the two segments

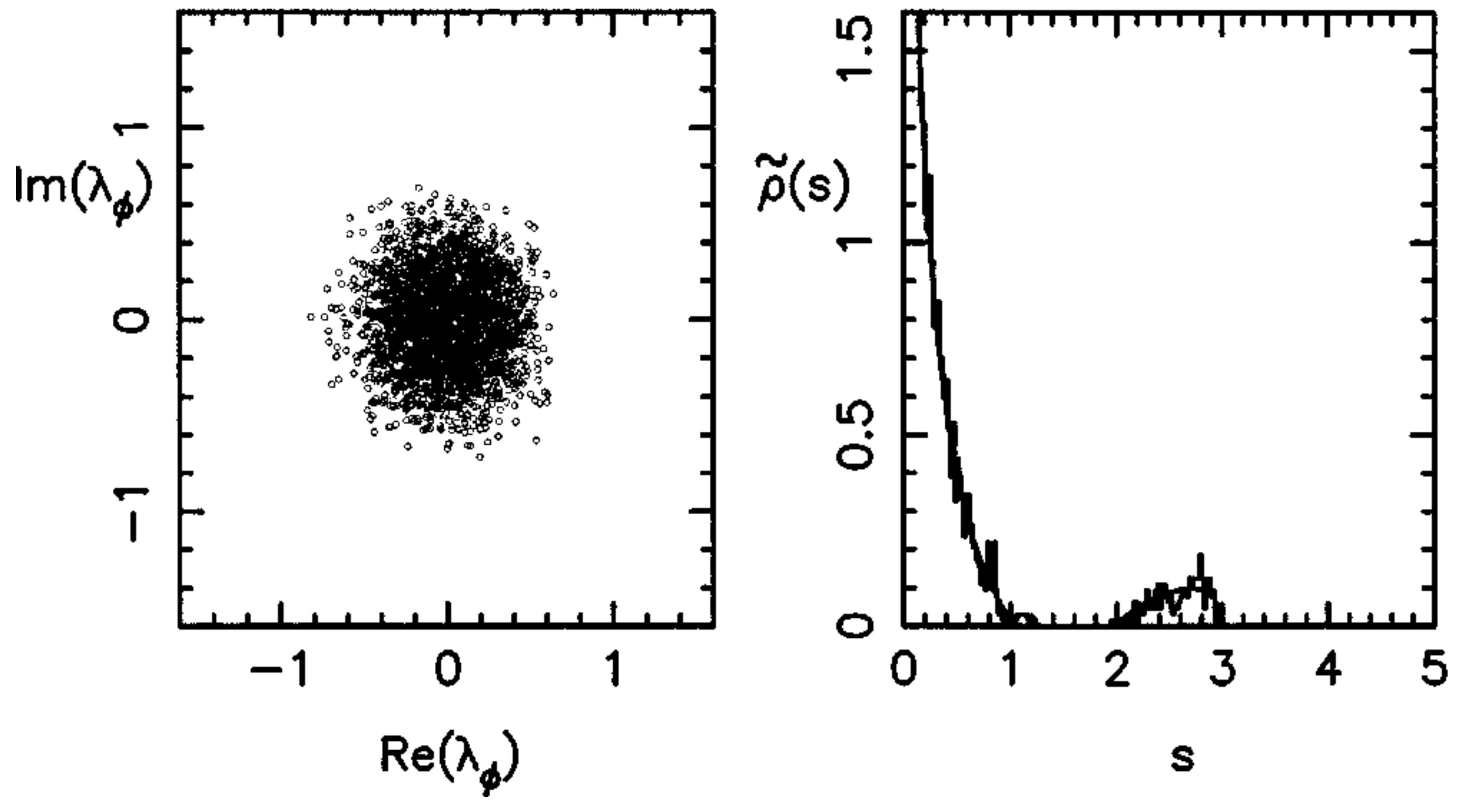
$[0, a] \cup [b, c]\,, 0 < a < b < c$

where a, b and c are determined according to the methods of JMP**42**(2001)5718

Comment: the segment edge c here is not to be confused with the coefficient in front of the penalizing commutator squared term

As was demonstrated in JMP**42**(2001)5718 by MC simulations, in the absence of the penalizing term (c=0) and for $g<0$, the complex eigenvalues of ϕ are always supported in a disk centered at the origin, despite the condensation of the eigenvalues of Λ^2 in two disjoint segments, confirming the SRT in these cases:

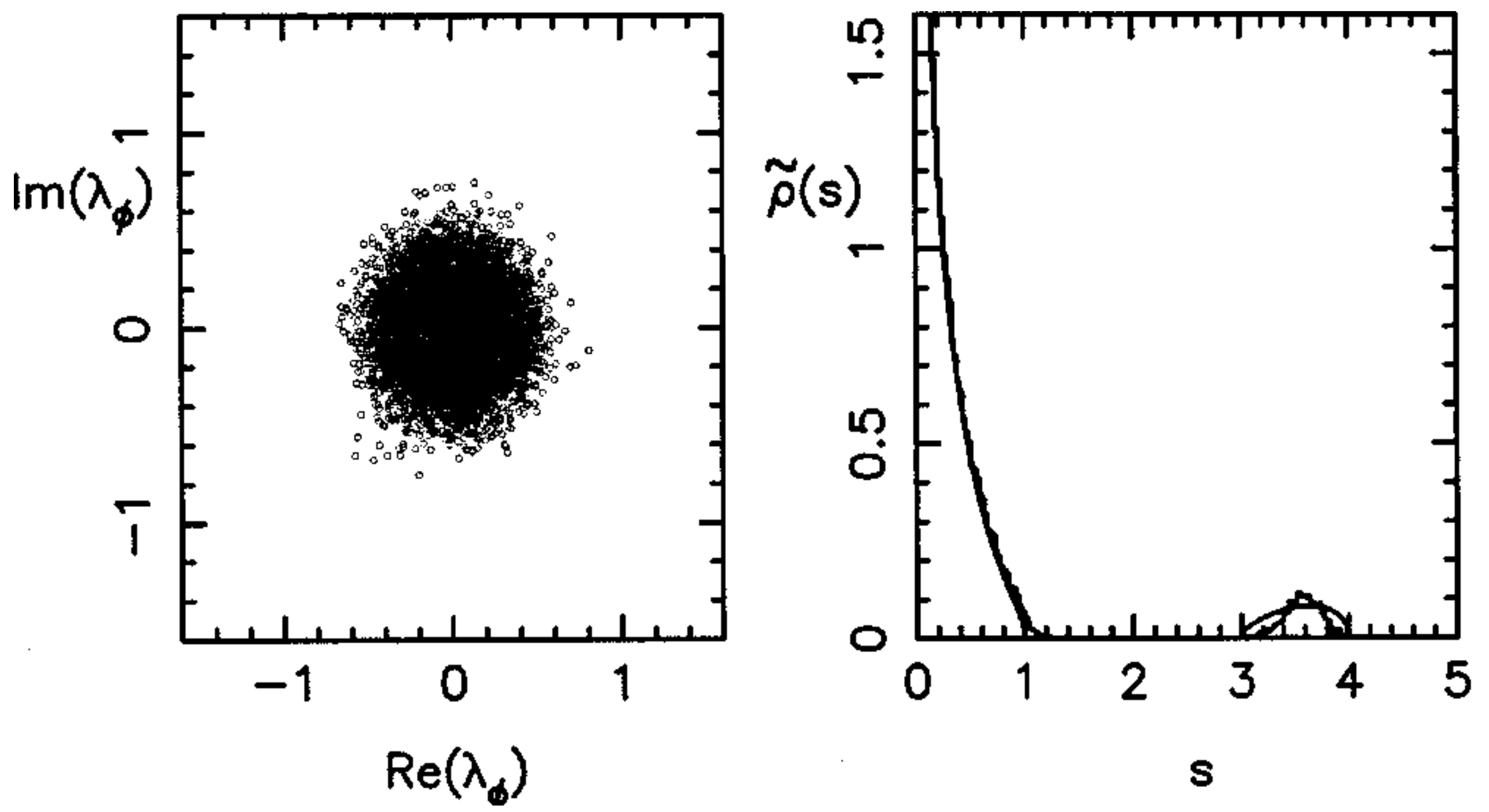
$$V(\phi^\dagger\phi)=m^2\phi^\dagger\phi+\frac{g}{2}(\phi^\dagger\phi)^2+\frac{u}{3}(\phi^\dagger\phi)^3$$



$N=32$

$m^2=7.372, g=-6.116, u=1.372$

$a=1, b=2, c=3: 0\leq s\leq 1\cup 2\leq s\leq 3$



$N=32$

$m^2=6.403, g=-4.184, u=0.713$

$a=1, b=3, c=4: 0\leq s\leq 1\cup 3\leq s\leq 4$

Turning the penalty term: MC simulations of the flow model

Notational comments: in the following plots λ stands for the radial variable r

$P(\lambda)$ stands for the radial density of eigenvalues $\rho(r)$ of ϕ

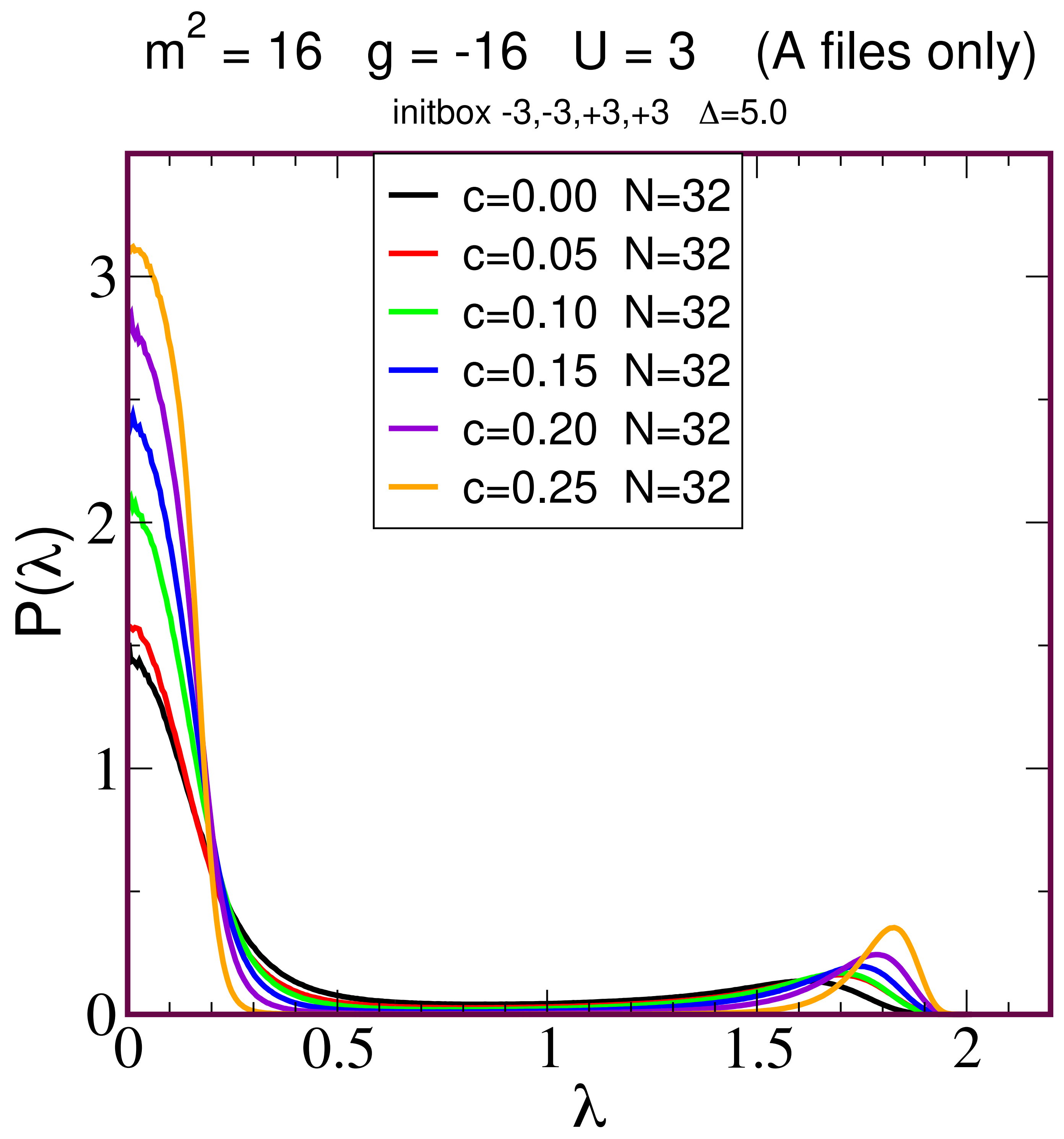


Figure 1. Eigenvalue distribution $\mathcal{P}(\lambda)$ for different c. As c increases, the distribution evolves into two disjoint regions of support.

logarithmic scale

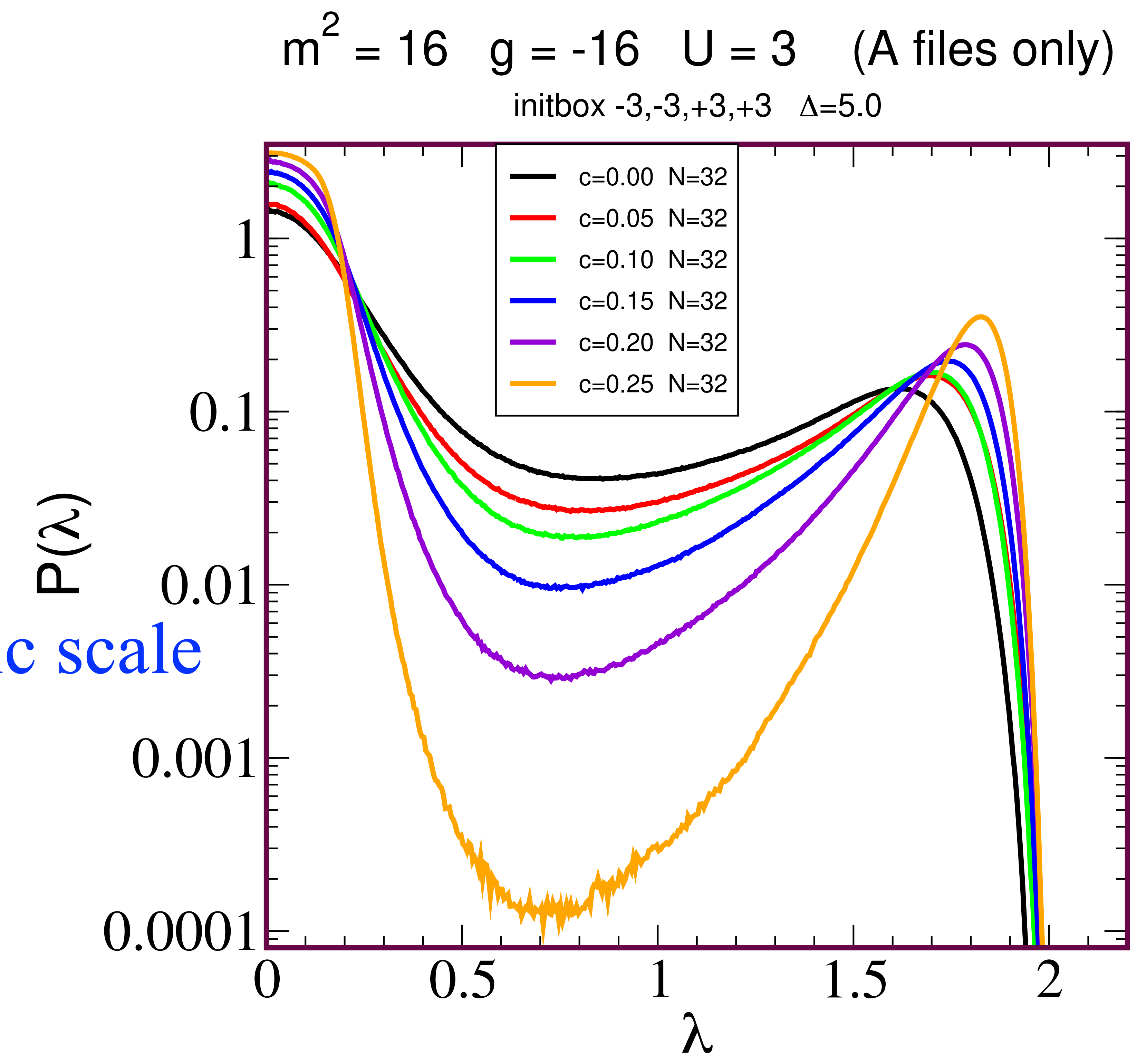


Figure 2. Eigenvalue distribution $\mathcal{P}(\lambda)$ for different c. As c increases, the distribution evolves into two disjoint regions of support.

formation of disjoint concentric disk and annulus.
For small c, the radial eigenvalue density is non-zero for all $r \lesssim 1.9$

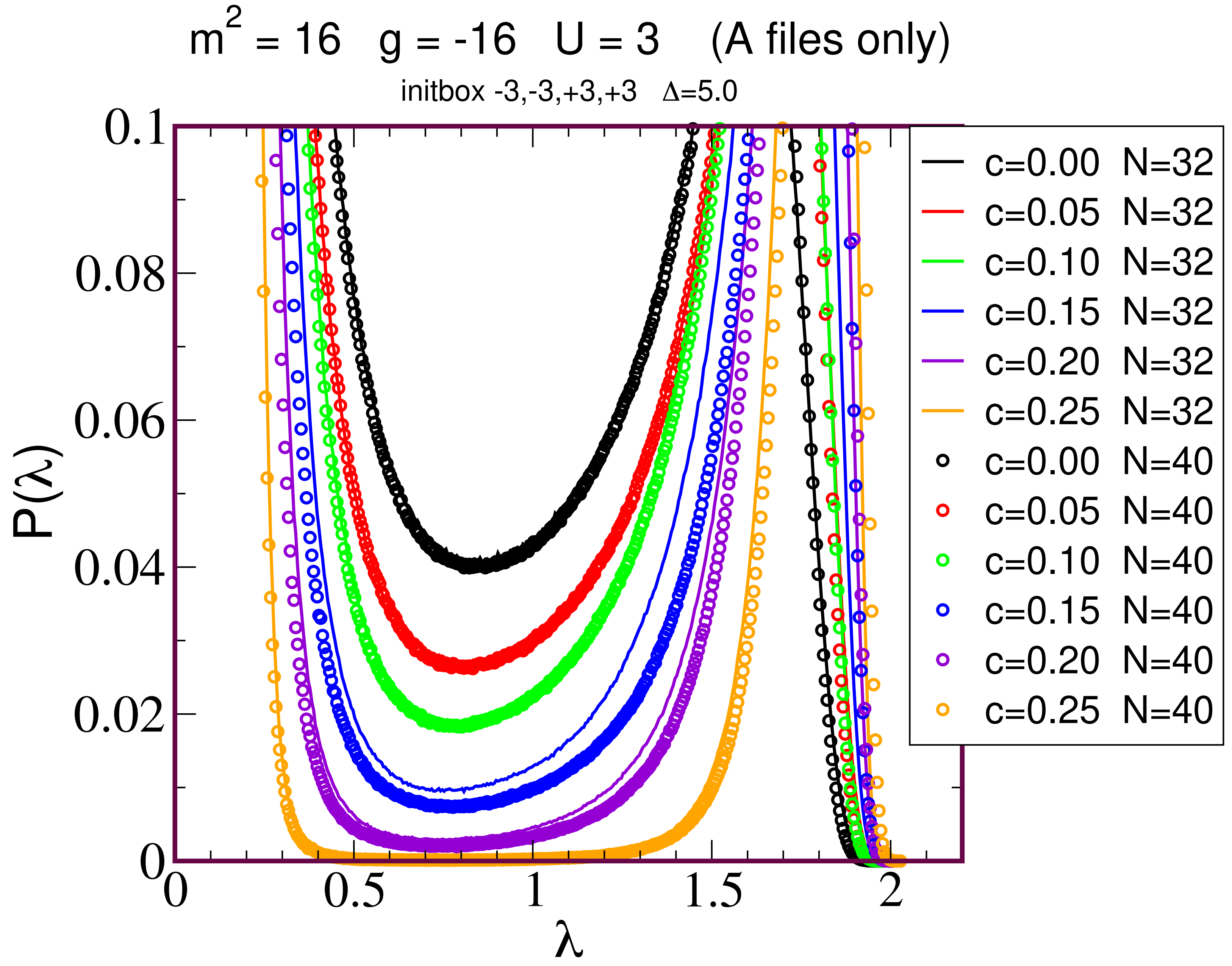


Figure 3. Zoom-in of the eigenvalue distribution $\mathcal{P}(\lambda)$ in the region of the ring-annulus gap. For small c the distribution is independent of matrix size N, but for larger c the value in the gap decreases with N.

almost

finite size (N) effects: the radial density decreases with N for c in the range $0.15 \lesssim c \lesssim 0.20$

the critical c_* should go down as one extrapolates to $N \to \infty$

Determining the critical penalty coefficient c_* at which SRT breaks down

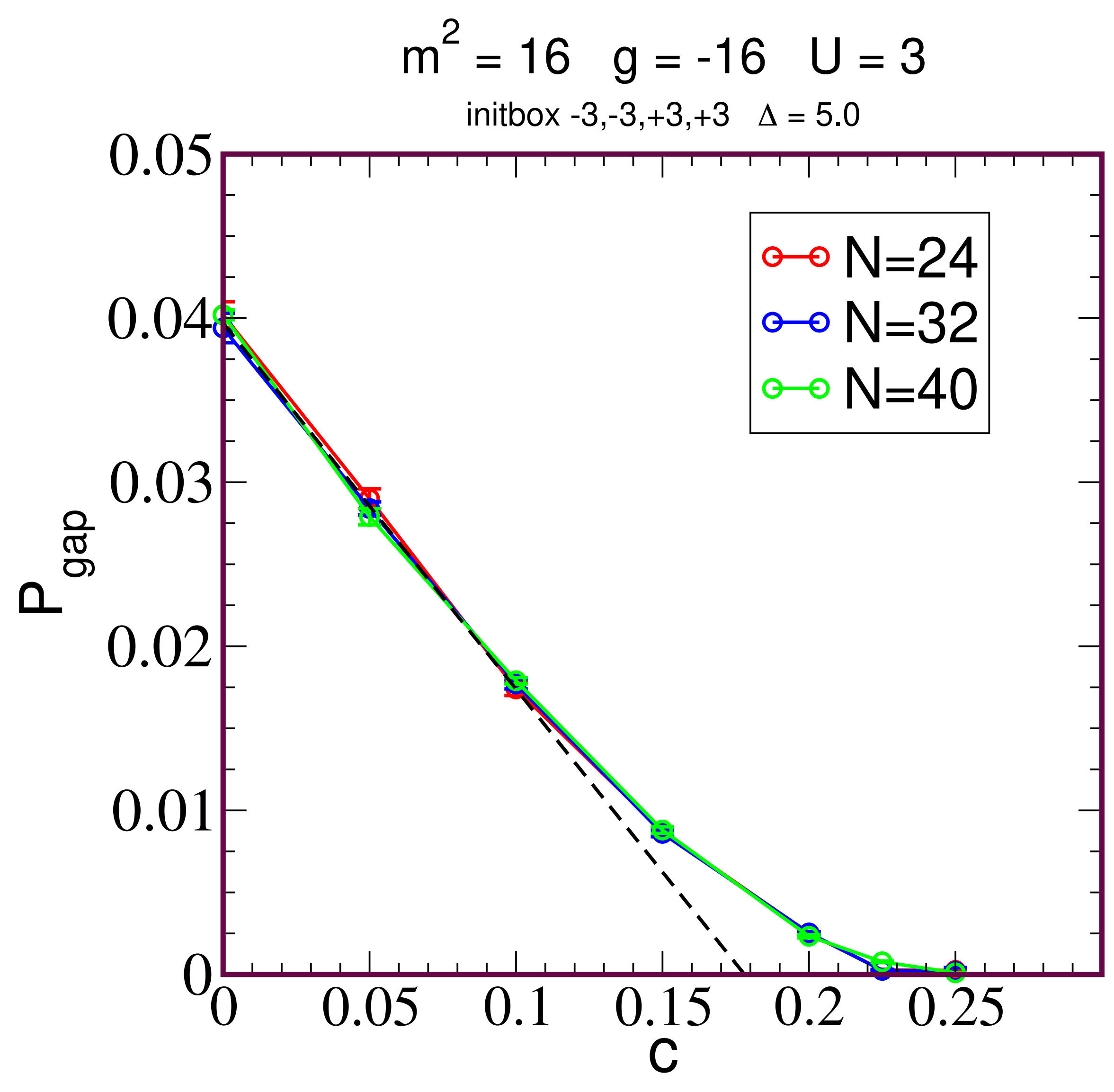


Figure 4. Plot of the minimum value of the eigenvalue distribution $\mathcal{P}(\lambda)$ in a window between the disk and annulus. The minimum value goes to zero at $c_* \sim 0.22 \pm 0.02$. Assuming the gap onsets linearly at the critical value c_* yields an estimate for the position of the transition to a ring+annulus phase $\tilde{c}_* \sim 0.18 \pm 0.02$.

starts decreasing linearly with c, then curves upward

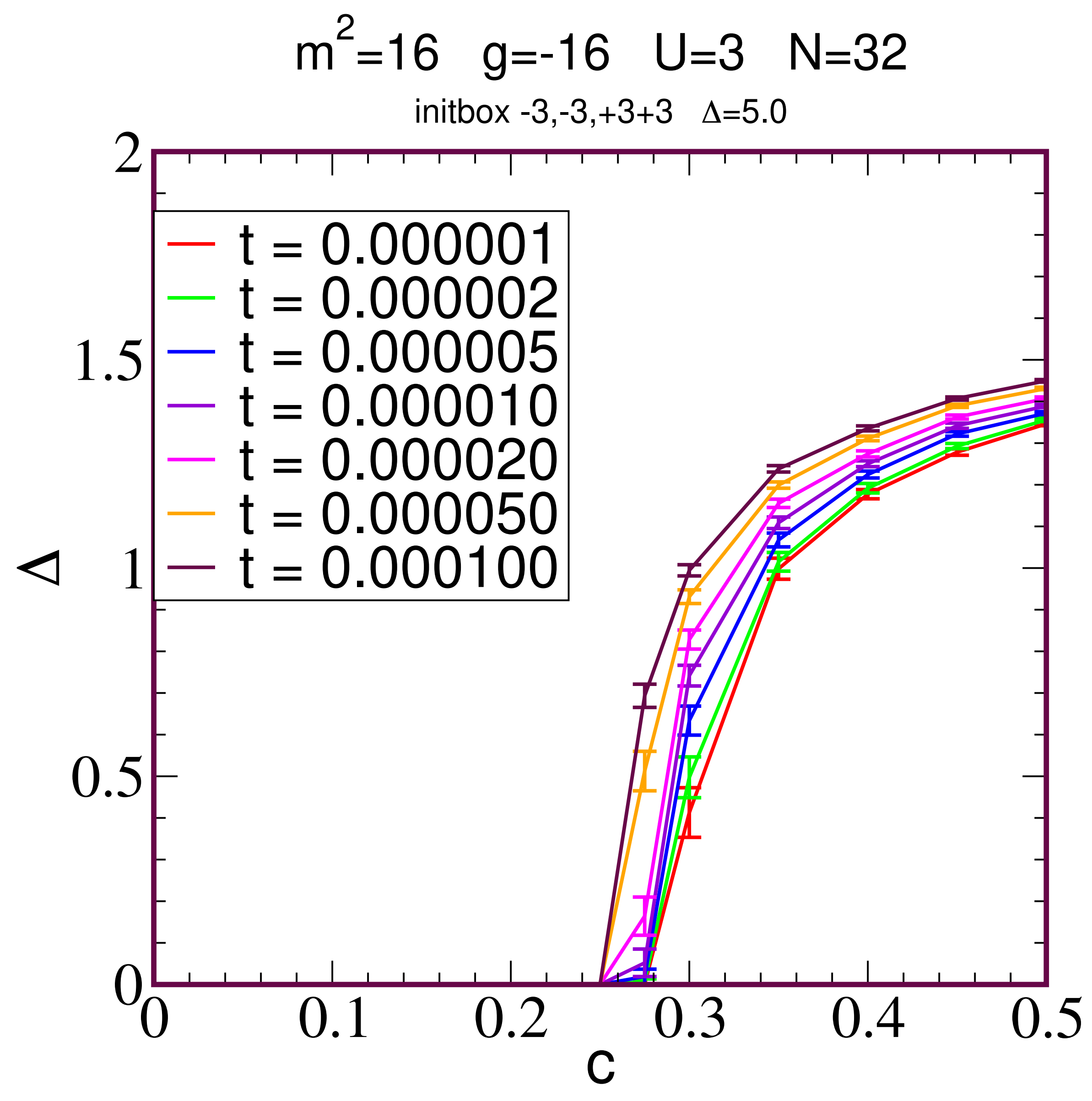


Figure 5. Plot of the size of the gap Δ of the eigenvalue density in the region between disk and annulus. The gap is defined by counting the number of bins whose density of states is less than some threshold t and multiplying by the bin width. This way of looking at the data suggests $c_* \sim 0.25 \pm 0.02$. Go

c_* is rather small. The SRT is fragile with respect to this flow

not easy to pin-point numerically

The Gas of Singular Values in the Flow Model

(A second topic not directly related to fragility of the SRT under the flow)

$$\phi = U\Lambda V\,, \quad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)\,, \quad W = VU$$

$$\mathrm{tr}[\phi, \phi^\dagger]^2 = 2\mathrm{tr}[(\phi^\dagger\phi)^2 - \phi^2\phi^{\dagger 2}] = 2\mathrm{tr}(\Lambda^4 - \Lambda^2 W^\dagger \Lambda^2 W)$$

$$\mathrm{tr}\left(V(\phi^\dagger\phi) + \frac{c}{4}[\phi, \phi^\dagger]^2\right) = \sum_{i=1}^{N}\left(V(\lambda_i^2) + \frac{c}{2}\lambda_i^4\right) - \frac{c}{2}\mathrm{tr}(\Lambda^2 W^\dagger \Lambda^2 W)$$

$$P_c(\phi, \phi^\dagger) = \frac{1}{\mathcal{Z}_c}\, e^{-N\mathrm{tr}\left(V(\phi^\dagger\phi) + \frac{c}{4}[\phi,\phi^\dagger]^2\right)}$$

$$\mathcal{Z}_c = \int d\phi d\phi^\dagger e^{-N\mathrm{tr}\left(V(\phi^\dagger\phi)+\frac{c}{4}[\phi,\phi^\dagger]^2\right)}$$

integration measure (in `polar coordinates') $d\phi d\phi^\dagger = \prod_{i\alpha} d\mathrm{Re}(\phi_{i\alpha})\, d\mathrm{Im}(\phi_{i\alpha}) = d\mu(U)d\mu(V)\Delta^2(\Lambda^2)\prod_k \lambda_k d\lambda_k$

(suppressed division by the volume of $U(1)^N = (2\pi)^N$)

(recall also that $\lambda_k \geq 0$)

Haar measure over unitary group $d\mu(U), d\mu(V)$

Vandermonde determinant $\Delta(\Lambda^2) = \prod_{i>j}(\lambda_i^2 - \lambda_j^2)$

homogeneity of the Haar measure $d\mu(V) = d\mu(VU) = d\mu(W)$

$$\therefore\, d\mu(U)d\mu(V) = d\mu(U)d\mu(W)$$

obtain

$$\mathcal{Z}_c = \mathrm{Vol}\left(\frac{U(N)}{U(1)^N}\right)\int_0^\infty \left(\prod_k \lambda_k d\lambda_k\right)\Delta(\Lambda^2)^2 e^{-N\sum_i\left(V(\lambda_i^2)+\frac{c}{2}\lambda_i^4\right)}\int_{U(N)} d\mu(W)e^{N\frac{c}{2}\mathrm{tr}(\Lambda^2W^\dagger\Lambda^2W)}$$

the last integral is an Itzykson-Zuber integral: $\displaystyle\int_{U(N)} d\mu(W)e^{N\frac{c}{2}\mathrm{tr}(\Lambda^2W^\dagger\Lambda^2W)} = \frac{C'}{c^{N(N-1)/2}}\frac{\det_{ij}e^{N\frac{c}{2}\lambda_i^2\lambda_j^2}}{\Delta^2(\Lambda^2)}$

where C' is a normalization constant independent of c making sure that in the limit $c\to 0$

the IZ integral reproduces $\mathrm{Vol}\,(U(N))$

Putting everything together, the Vandermondes in the denominator knock out those in the denominator, ending up with

$$\therefore\ \mathcal{Z}_c = \frac{2^NC}{c^{N(N-1)/2}}\int_0^\infty \left(\prod_k \lambda_k d\lambda_k\right)e^{-N\sum_i\left(V(\lambda_i^2)+\frac{c}{2}\lambda_i^4\right)}\det_{ij}e^{\frac{Nc}{2}\lambda_i^2\lambda_j^2}$$

2^NC a combined constant

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generic IZ integral $\displaystyle\int_{U(N)} d\mu(W)e^{\mathrm{tr}(AW^\dagger BW)} = \mathrm{const.}\frac{\det_{ij}e^{a_ib_j}}{\Delta(A)\Delta(B)}$ $A=\mathrm{diag}(a_1,a_2,\ldots,a_N)\,,B=\mathrm{diag}(b_1,b_2,\ldots,b_N)$

change of variables $\lambda_i^2 = x_i$

$$\therefore \mathcal{Z}_c = \frac{C}{c^{N(N-1)/2}} \int_0^\infty \prod_k dx_k e^{-N\sum_i \left(V(x_i) + \frac{c}{2}x_i^2\right)} \det_{ij} e^{\frac{Nc}{2}x_i x_j}$$

$$= \frac{C}{c^{N(N-1)/2}} \int_0^\infty \prod_k dx_k e^{-N\sum_i V(x_i)} \det_{ij}\left(e^{-\frac{Nc}{4}(x_i - x_j)^2}\right)$$

$$= \frac{C}{c^{N(N-1)/2}} \int_0^\infty \prod_k dx_k \det_{ij}\left(e^{-\frac{N}{2}V(x_i)} e^{-\frac{Nc}{4}(x_i - x_j)^2} e^{-\frac{N}{2}V(x_j)}\right)$$

Non-interacting Fermi Gas on the Positive Half-Line

$$|x_1, x_2, \ldots, x_N\rangle_A = \frac{1}{\sqrt{N!}} \sum_{\sigma \in S_N} (-1)^\sigma |x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(N)}\rangle$$

anti-symmetrized N-position ket (similarly for bra)

$$\langle x_i | e^{-\beta \hat{h}_{eff}} | x_j \rangle = e^{-\frac{N}{2} V(x_i)} e^{-\frac{Nc}{4}(x_i - x_j)^2} e^{-\frac{N}{2} V(x_j)}$$

effective one-particle hamiltonian (later explicit example)

<u>diagonal matrix element</u> of the density matrix of N-noninteracting fermions on the positive half-line

$$\det_{ij}\left(e^{-\frac{N}{2} V(x_i)} e^{-\frac{Nc}{4}(x_i - x_j)^2} e^{-\frac{N}{2} V(x_j)}\right) = {}_A\langle x_1, x_2, \ldots, x_N | \prod_k e^{-\beta \hat{h}_{eff}(k)} | x_1, x_2, \ldots, x_N \rangle_A$$

Two Common Heat Kernels on the Half-Line

Dirichlet b.c. at the origin (sum over all random walkers which avoid the origin)

$$G_D(x,y;t) = \langle x|e^{-t\hat{p}_D^2}|y\rangle = \frac{1}{\sqrt{4\pi t}}\left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}}\right)$$

Neumann b.c. at the origin (sum over all random walkers which are reflected back at the origin)

$$G_N(x,y;t) = \langle x|e^{-t\hat{p}_N^2}|y\rangle = \frac{1}{\sqrt{4\pi t}}\left(e^{-\frac{(x-y)^2}{4t}} + e^{-\frac{(x+y)^2}{4t}}\right)$$

our kernel is an even mixture of the two:

$$G_{DN}(x,y;t) = \frac{1}{2}\left(G_D(x,y;t) + G_N(x,y;t)\right) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}$$

Our density matrix describes a mixture of Dirichlet and Neumann particles. There are two effective one-particle hamiltonians: $\hat{h}^D_{eff}\,, \hat{h}^N_{eff}$

after a little bit of thinking we see that we can simplify the problem by extending it to the full line in a symmetric way:

$$V(x) \to V_s(x) = V_s(-x)$$

which would lead to a parity-symmetric effective one-particle effective hamiltonian $\hat{h}_s$ defined by

$$\langle x|e^{-\beta \hat{h}_s}|y\rangle = \langle x|e^{-\frac{1}{2}NV_s(\hat{x})}e^{-\beta\hat{p}^2}e^{-\frac{1}{2}NV_s(\hat{x})}|y\rangle$$

and its eigenfunctions are of definite parity.

The effective Dirichlet and Neumann hamiltonians $\hat{h}^D_{eff}, \hat{h}^N_{eff}$ are simply projections of $\hat{h}_s$, respectively to its odd and even parity sectors (followed by restriction to the positive half-line)

In summary: work with the non-interacting N-fermion system, subjected to the parity symmetric one-particle hamiltonian $\hat{h}_s$ compute the diagonal matrix element of the N-fermion density matrix, and restrict to the positive half-line, normalization factor to ensure proper normalization

Explicit computation of $\hat{h}_s$ in the general case is impossible, but in one case there is explicit construction.

Change of Level Statistics among Singular Values under the Flow

in order to demonstrate this phenomenon, the explicit form of $\hat{h}_s$ is not required.

in order to make the point it is enough to consider level statistics at the two endpoints of the flow:

$c = 0$ & $c \to \infty$

at $c = 0$ there is no commutator term, no Itzykson-Zuber integral

$$\mathcal{Z}_0 = \mathrm{Vol}\left(\frac{U(N)}{U(1)^N}\right)\int_0^\infty \left(\prod_k \lambda_k d\lambda_k\right)\Delta(\Lambda^2)^2 e^{-N\sum_i V(\lambda_i^2)} = C\int_0^\infty \prod_k dx_k \Delta^2(X) e^{-N\sum_i V(x_i)}$$

$\Delta^2(X) = \prod_{i>j}(x_i - x_j)^2$ familiar from RMT of complex-hermitian matrices (Dyson index $\beta = 2$)

the probability density to find two coinciding $x_i's$ clearly vanishes.

The system has Wigner-Dyson level statistics of $\beta = 2$ type, which means, in particular that the $x_i's$ repel each other so that the probability of having very small nearest-neighbor spacing s vanishes like $s^\beta = s^2$

as $\quad c \to \infty \quad$ in $\quad \mathcal{Z}_c = \dfrac{C}{c^{N(N-1)/2}} \int_0^\infty \prod_k dx_k e^{-N\sum_i V(x_i)} \det_{ij}\left(e^{-\frac{Nc}{4}(x_i - x_j)^2}\right)$

we see that in order to be correlated, any two particles have to be within a very small distance $|x_i - x_j| \sim \dfrac{1}{\sqrt{Nc}}$

on this very short distance scale, they repel: because of the determinant, the probability of having two particles at the same point vanishes.

that two particles get this close is a rare event, and the N particles will almost always be father away, and therefore uncorrelated.

the nearest-neighbor spacing distribution of strictly uncorrelated particles is Poissonian $P(s) = e^{-s}$

a concrete construction of $\hat{h}_s$

$$V(\phi^\dagger\phi) = (\phi^\dagger\phi)^2$$

$$V(x) = V_s(x) = x^2$$

$$e^{-\frac{N}{2}V(x_i)}e^{-\frac{Nc}{4}(x_i-x_j)^2}e^{-\frac{N}{2}V(x_j)} = e^{-\frac{N}{2}(1+\frac{c}{2})(x_i^2+x_j^2)+\frac{Nc}{2}x_ix_j}$$

compare this to the Feynman propagator of the harmonic oscillator (in Euclidean time):

$$\langle x_i|\exp\left[-\beta\left(\frac{\hat{p}^2}{2m}+\frac{m\omega^2\hat{x}^2}{2}\right)\right]x_j\rangle = C\exp\left[-\frac{m\omega}{2\hbar}\left(\coth(\beta\hbar\omega)(x_i^2+x_j^2)-\frac{2x_ix_j}{\sinh(\beta\hbar\omega)}\right)\right]$$

$$C = \sqrt{\frac{m\omega}{2\pi\hbar\sinh(\beta\hbar\omega)}}$$

leading to the identifications $\frac{m\omega}{\hbar}\coth(\beta\hbar\omega) = N(1+\frac{c}{2})$ and $\frac{m\omega}{\hbar\sinh(\beta\hbar\omega)} = \frac{Nc}{2}$

$$\therefore \cosh(\beta\hbar\omega) = 1+\frac{2}{c}\,, \quad \frac{m\omega}{\hbar} = N\sqrt{1+c}$$

this identification was made originally in the paper

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Generalized Ensemble of Random Matrices

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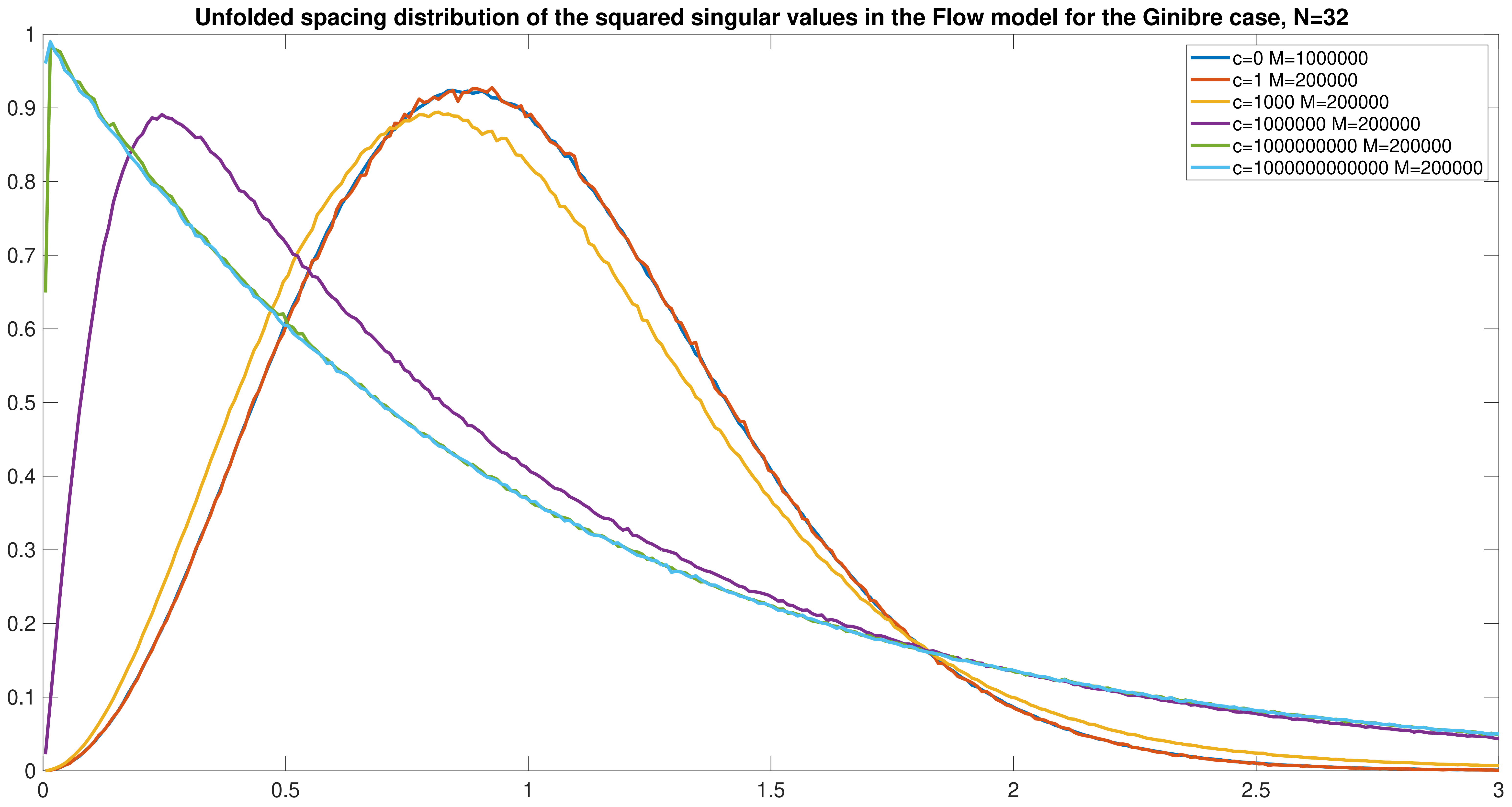
A random matrix ensemble incorporating both Gaussian unitary ensemble and Poisson level statistics while respecting $U(N)$ invariance is proposed and shown to be equivalent to a system of noninteracting, confined, one-dimensional fermions at finite temperature.

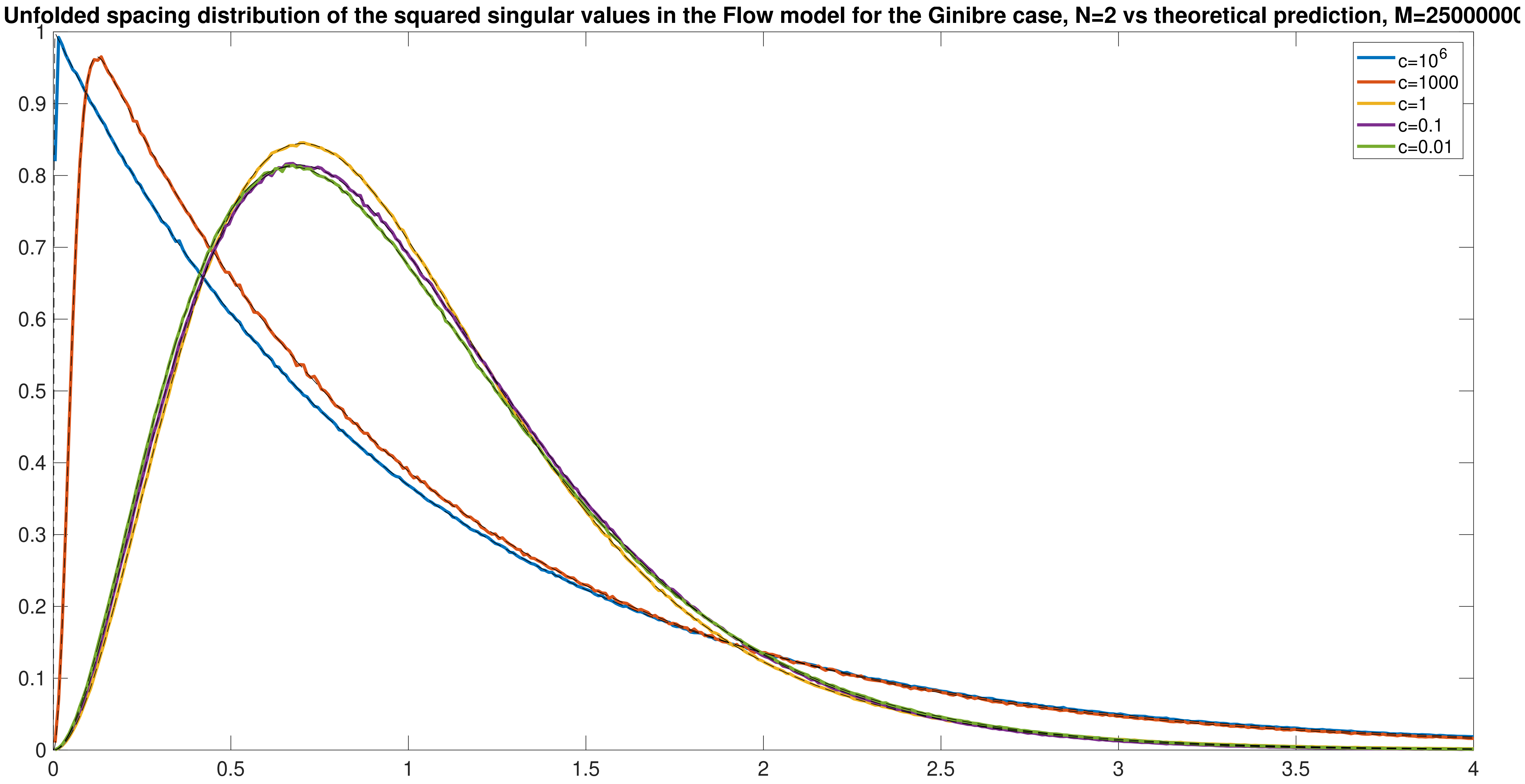
Ginibre's model $\quad V(x) = x$



N=2 Ginibre - exact calculation (analogously to the `Wigner Surmise')

$$P(s) = \frac{C'}{c} e^{-2s}(1 - e^{-cs^2})$$

$$s \gg \frac{1}{\sqrt{c}} \Longrightarrow P(s) \simeq \frac{C'}{c} e^{-2s}$$



Thanks for your attention!