

Systematic classification of solvable potentials within and outside the Natanzon class

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1. Exactly solvable potentials in general

Why are they important... especially if they are **complex**?

How are they generated?

Variable transformation $z(x)$: Schrödinger eq. \implies diff. eq. of special function $F(z)$

SUSYQM: Known solvable potential $a \implies$ new solvable potentials

How are they classified?

Using the special function $F(z)$, the **variable transformation** $z(x)$ and **SUSYQM**

2. Natanzon potentials: the (confluent) hypergeometric differential equation

Adaptating the techniques ${}_2F_1(a, b; c; z)$ and ${}_1F_1(a; c; z)$

Bound states: **Jacobi and generalized Laguerre polynomials**

Shape invariance

3. Beyond the Natanzon class: Heun type differential equations

Example 1: The rationally extended harmonic oscillator: SUSY partner of the HO

Example 2: The sextic oscillator: introduced as QES potential

1. Exactly solvable potentials in general

Milestones of generating solvable potentials

1940: Factorization method	<i>Schrödinger</i>
1951: Systematic application of the factorization method	<i>Infeld and Hull</i>
1962: A variable transformation method	<i>Bhattacharjie and Sudarshan</i>
1971: Systematic application of the transformation method	<i>Natanzon</i>
1981: SUSYQM: a reformulation of the factorization method	<i>Witten</i>

Note the [periodicity](#) of $\simeq 11$ years

Other examples for phenomena with $\simeq 11$ year periodic activity maxima

A yellow dwarf



Other examples for phenomena with $\simeq 11$ year periodic activity maxima

A yellow dwarf

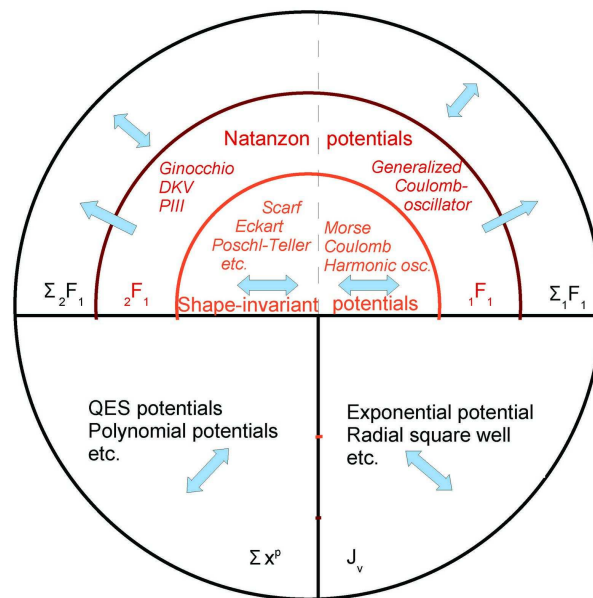
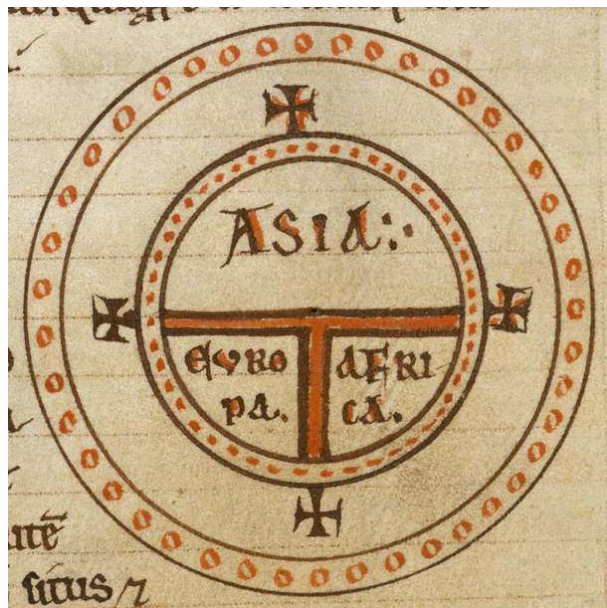


A red giant

1945 Berlin
1956 Budapest
1968 Prague
1979 Kabul
1991 Moscow
...



The world map of solvable potentials



Any more continents or islands?

The main territories in the map

${}_2F_1$: **Natanzon class**, solved by ${}_2F_1$ in general, by $P_n^{(\alpha,\beta)}(z)$ for bound states

${}_1F_1$: **Natanzon confluent class**, solved by ${}_1F_1$ in general, by $L_n^{(\alpha)}(z)$ for bound states

Shape-invariant: Natanzon (confluent) subclass, **closed** under a SUSY transformation

$\Sigma_2 F_1$ and $\Sigma_1 F_1$: solutions in terms of the **linear combination** of several
(confluent) hypergeometric functions

Non-SI SUSY partners of Natanzon (confluent) potentials

Potentials solved by **exceptional orthogonal polynomials**, a new type of SI

Solutions containing **both independent solutions** gen. Woods–Saxon

J_ν : potentials solved by **Bessel functions**

Σx^p : **Quasi-exactly** solvable potentials: exact solutions up to a finite n

The variable transformation method

Bhattacharjie and Sudarshan 1962

Schrödinger eq. \implies differential equation of special function F

$$\frac{d^2\psi}{dx^2} + (E - V(x))\psi(x) = 0 \qquad \text{insert} \qquad \psi(x) = f(x)F(\mathbf{z}(x))$$

and compare with

$$\frac{d^2F}{d\mathbf{z}^2} + \textcolor{blue}{Q}(\mathbf{z})\frac{dF}{d\mathbf{z}} + \textcolor{blue}{R}(\mathbf{z})F(\mathbf{z}) = 0$$

to get

$$E - V(x) = \frac{\mathbf{z}'''(x)}{2\mathbf{z}'(x)} - \frac{3}{4}\left(\frac{\mathbf{z}''(x)}{\mathbf{z}'(x)}\right)^2 + (\mathbf{z}'(x))^2\left(\textcolor{blue}{R}(\mathbf{z}(x)) - \frac{1}{2}\frac{d\textcolor{blue}{Q}(\mathbf{z})}{d\mathbf{z}} - \frac{1}{4}\textcolor{blue}{Q}^2(\mathbf{z}(x))\right) .$$

Schwartzian derivative terms E and the main potential terms

Connection to
SUSYQM:

$$W(x) = -\frac{1}{2}Q(z(x))z'(x) + \frac{z''(x)}{2z'(x)}$$

The solutions are

$$\psi(x) \sim (\mathbf{z}'(x))^{-\frac{1}{2}} \exp \left(\frac{1}{2} \int^{\mathbf{z}(x)} Q(\mathbf{z}) d\mathbf{z} \right) F(\mathbf{z}(x)) .$$

The yet unknown $\mathbf{z}(x)$ can be obtained from a differential equation

$$\left(\frac{d\mathbf{z}}{dx} \right)^2 \phi(\mathbf{z}) = C ,$$

by direct integration

$$\int \phi^{1/2}(\mathbf{z}) d\mathbf{z} = C^{1/2} x + \gamma .$$

γ : integration constant, coordinate shift

This is to generate a **constant term** on the r.h.s. of $E - V(x) = \dots$

The special function F (i.e. Q and R) and the variable transformation $\mathbf{z}(x)$

determines everything

Note: sometimes only $x(\mathbf{z})$ can be determined \implies implicit potentials

2. Natanzon potentials: the (confluent) hypergeometric differential equation

Natanzon-class potentials

Natanzon 1971

Employ the method to the hypergeometric function ${}_2F_1(a, b; c; z)$

Or the confluent hypergeometric function ${}_1F_1(a; c; z)$

For bound states these functions reduce to **orthogonal polynomials**:

$${}_2F_1(a, b; c; z) \implies P_n^{(\alpha, \beta)}(1 - 2z) \quad \text{for } a = -n \text{ or } b = -n$$

Jacobi polynomial

$${}_1F_1(a; c; z) \implies L_n^{(\alpha)}(z) \quad \text{for } a = -n$$

Generalized Laguerre polynomial

Jacobi polynomials sometimes represent a better choice:

They can be adapted better to \mathcal{PT} -symmetric QM:

Their argument exhibits \mathcal{PT} symmetry

Apply the method to the Jacobi polynomials: $F(z) = P_n^{(\alpha, \beta)}(z)$

$$\begin{aligned}
 E - V(x) = & \frac{z'''(x)}{2z'(x)} - \frac{3}{4} \left(\frac{z''(x)}{z'(x)} \right)^2 + \frac{(z'(x))^2}{1 - z^2(x)} \left(n + \frac{\alpha + \beta}{2} \right) \left(n + \frac{\alpha + \beta}{2} + 1 \right) \\
 & + \frac{(z'(x))^2}{(1 - z^2(x))^2} \left[1 - \left(\frac{\alpha + \beta}{2} \right)^2 - \left(\frac{\alpha - \beta}{2} \right)^2 \right] \\
 & - \frac{2z(x)(z'(x))^2}{(1 - z^2(x))^2} \left(\frac{\alpha + \beta}{2} \right) \left(\frac{\alpha - \beta}{2} \right) .
 \end{aligned}$$

The solutions are

$$\psi(x) \sim (z'(x))^{-\frac{1}{2}} (1 + z(x))^{\frac{\beta+1}{2}} (1 - z(x))^{\frac{\alpha+1}{2}} P_n^{(\alpha, \beta)}(z(x)) .$$

The yet unknown $\mathbf{z}(x)$ can be obtained from a differential equation

$$\left(\frac{dz}{dx} \right)^2 \phi(z) \equiv \left(\frac{dz}{dx} \right)^2 \frac{p_{\text{I}}(1-z^2) + p_{\text{II}} + p_{\text{III}}z}{(1-z^2)^2} = C .$$

by direct integration

$$\int \phi^{1/2}(\mathbf{z}) d\mathbf{z} = C^{1/2} x + \gamma .$$

C : scale parameter, γ : integration constant, coordinate shift

Separating the constant E term, the potential is

$$V(x) = -\frac{z'''(x)}{2z'(x)} + \frac{3}{4} \left(\frac{z''(x)}{z'(x)} \right)^2 + \frac{C}{\phi(z)} [s_I(1 - z^2(x)) + s_{II} + s_{III}z(x)] .$$

The solutions are

$$\psi(x) \sim [\phi(z(x))]^{\frac{1}{4}} (1 + z(x))^{\frac{\beta}{2}} (1 - z(x))^{\frac{\alpha}{2}} P_n^{(\alpha, \beta)}(z(x)) .$$

$$(n + \frac{1}{2} + \omega)^2 - \frac{1}{4} + s_I - p_I \frac{E_n}{C} = 0$$

$$(1 - \omega^2 - \rho^2) + s_{II} - p_{II} \frac{E_n}{C} = 0$$

$$-2\omega\rho + s_{III} - p_{III} \frac{E_n}{C} = 0$$

$$\omega = (\alpha + \beta)/2 \text{ and } \rho = (\alpha - \beta)/2$$

Solving the problem: chose $p_i \implies$ get $z(x) \implies$ express $E_n \implies$ get $V(x)$

About the origin of the parameters:

s_i : from the parameters of the polynomial α and β

p_i : (plus also C and ϵ) from the variable transformation function $z(x)$

GL Int. J. Theor Phys. **54** (2015) 2724

What about the role of SUSYQM?

New potentials are generated from old ones

The solutions of the new potentials are obtained from those of the old ones:

$$\psi_+(x) = \left(\frac{d}{dx} + W(x) \right) \psi_-(x)$$

If $\psi_-(x)$ contains $P_n^{(\alpha,\beta)}(z(x))$ then ...

... $\psi_+(x)$ contains $P_n^{(\alpha,\beta)}(z(x))$ AND $P_n^{(\alpha,\beta)}(z(x))'$

...so it is a linear combination of two Jacobi polynomials

BUT sometimes **recursion relations** help to restore the original structure

Shape-invariant potentials: they correspond to simple choice of $z(x)$, i.e. p_i

The list of (real) **shape-invariant** potentials ($a = 1$, $C = \pm 1$)

$(z')^2 =$ (Class)	$V(x)$	$x \in$	Name
$C(1 - z^2)$ (PI)	$(B^2 - A^2 - A)\text{sech}^2(x) + B(2A + 1)\text{sech}(x) \tanh(x)$ $(B^2 + A^2 + A)\text{cosech}^2(x) - B(2A + 1)\text{cosech}(x) \coth(x)$ $(B^2 + A^2 - A)\text{cosec}^2(ax) - B(2A - 1)\text{cosec}(x) \cot(x)$ $A(A - 1) \sec^2(x) + B(B - 1)\text{cosec}^2(x)$ $-A(A + 1)\text{sech}^2(x) + B(B - 1)\text{cosech}^2(x)$	$(-\infty, \infty)$ $[0, \infty)$ $[0, \pi]$ $[0, \pi/2]$ $[0, \infty)$	Scarf II gen. Pöschl–Teller Scarf I Pöschl–Teller I Pöschl–Teller II
$C(1 - z^2)^2$ (PII)	$-A(A + 1)\text{sech}^2(x) + 2B \tanh(x)$ $A(A - 1)\text{cosech}^2(x) - 2B \coth(x)$ $A(A + 1)\text{cosec}^2(x) - 2B \cot(x)$	$(-\infty, \infty)$ $[0, \infty)$ $[0, \pi]$	Rosen–Morse II Eckart Rosen–Morse I
Cz (LI)	$\frac{1}{4}\omega^2 x^2 + \frac{l(l+1)}{x^2} - (l + \frac{3}{2})\omega$	$[0, \infty)$	3d harmonic oscillator
C (LII)	$\frac{e^4}{4(l+1)^2} - \frac{e^2}{x} + \frac{l(l+1)}{x^2}$	$[0, \infty)$	Coulomb
Cz^2 (LIII)	$A^2 - B(2A + 1) \exp(-x) + B^2 \exp(-2x)$	$(-\infty, \infty)$	Morse
C (HI)	$-\frac{1}{2}\omega + \frac{1}{4}\omega^2 x^2$	$(-\infty, \infty)$	1d harmonic oscillator

Obtained by selecting **certain single terms** on the right handside of $E - V(x) = \dots$

Constructing more general Natanzon-class potentials

Select **certain combinations** on the right handside of $E - V(x) = \dots$

$(z')^2 =$	$z(x)$	$F(z)$	$x \in$	Name	Ref.
$C(1 - z^2)^2 z^{-1}$	implicit	$P_n^{(\alpha,\beta)}(z)$	$(-\infty, \infty)$	sym. Ginocchio	Ginocchio 1984
$C(1 - z^2)^2 z^{-1}$	implicit	$P_n^{(\alpha,\beta)}(z)$	$[0, \infty)$	PIII	Lévai 1991
$C(z + \theta)$	implicit	$L_n^{(\alpha)}(z)$	$[0, \infty)$	gen. Coulomb	Lévai et al. 1993, 1998
$C(1 - z^2)(1 - z)z^{-1}$	implicit	$P_n^{(\alpha,\beta)}(z)$	$[0, \infty)$	WRL95	Williams et al. 1995
			confined	WRL95	Williams et al. 1995
$C(1 - z^2)^2 z^{-2}$	explicit	$P_n^{(\alpha,\beta)}(z)$	$(-\infty, \infty)$	DKV (PIV)	Dutt et al. 1995
$C(1 - z^2)^2 (z + \gamma)^{-2}$	implicit	$P_n^{(\alpha,\beta)}(z)$	$[0, \infty)$	WL03	Williams et al. 2003
$C(1 - z^2)^2 (\delta + 1 - z)^{-1}$	implicit	$P_n^{(\alpha,\beta)}(z)$	$(-\infty, \infty)$	L12	Lévai 2012

Most of these potentials are **weakly singular** at the finite boundaries

3. Potentials beyond the Natanzon class and the Heun type differential equations

Example 1: the rationally extended harmonic oscillator potential

Solved by the [X₁ type exceptional Laguerre polynomials](#) *Gomez-Ullate et al. 2010*

$$\hat{L}_n^{(\alpha)}(z) = -(\alpha + 1 + z)L_{n-1}^{(\alpha)}(z) + L_{n-2}^{(\alpha)}(z).$$

They satisfy a differential equation similar to generalized Laguerre polynomials

They form an orthogonal basis, but start with degree $\nu = n + 1 > 0$

Linear combinations of [two generalized Laguerre polynomials](#)

The potential:

Quesne 2008, Bagchi et al. 2009

$$\begin{aligned}\hat{V}(r) = & \frac{\omega^2}{4}r^2 + \frac{l(l+1)}{r^2} \\ & + \frac{4\omega}{2l+1+\omega r^2} - \frac{8\omega(2l+1)}{(2l+1+\omega r^2)^2}.\end{aligned}$$

Bound-state eigenfunctions:

$$\hat{\psi}_n(l; r) = \hat{C}_n \frac{r^{l+1}}{2l+1+\omega r^2} \exp\left(-\frac{\omega^2}{4}r^2\right) \hat{L}_{n+1}^{(l+\frac{1}{2})}\left(\frac{\omega}{2}r^2\right).$$

Energy eigenvalues:

$$\hat{E}_n = \omega\left(2n + l + \frac{3}{2}\right).$$

This potential is related to the harmonic oscillator potential by **SUSYQM**

It is **shape-invariant**

Take the harmonic oscillator potential and a SUSY transformation with

$$\chi(r) = r^{l+1} \exp\left(\frac{\omega^2}{4} r^2\right) (p + \omega r^2) .$$

and factorization energy

$$\epsilon = -\omega\left(l + \frac{7}{2}\right) < E_0^{(-)} .$$

$V_+(x)$ becomes the rationally extended harmonic oscillator potential

The two spectra are identical: **broken SUSY**

See *GL, T. Soltész, Eur. J. Phys.* **41** (2020) 025403 for a comprehensive discussion of the network of SUSY transformations between the harmonic oscillator and its rationally extended version (Laguerre)

See also the same for PI class potentials and their rational extension (Jacobi): *T. Soltész, L. F. Pethő, GL, Symmetry* **16** (2024) 174

Example 2: the sextic oscillator as a QES potential

Turbiner, Ushveridze

$$V(r) = \frac{(2s - 1/2)(2s - 3/2)}{r^2} + \left(b^2 - 4a(s + M + 1/2)\right)r^2 + 2abr^4 + a^2r^2$$

Solutions:

$$\psi(r) = Cr^{2s-1/2} \exp\left(-\frac{ar^4}{4} - \frac{br^2}{2}\right) P_M(r^2) .$$

$P_M(r^2)$: M 'th order polynomial

Its coefficients are determined by a [three-term recurrence relation](#)...

... which is terminated by a specific choice of the parameters

This **restricts** the generality of the potential

The general potential is solved in terms of an infinite power series.

There are infinite number of physical solutions, of which the **first $M+1$** can be obtained,
hence **QES**.

The energy eigenvalues are obtained by solving an algebraic equation of order $M + 1$.

Alternative approach to these beyond-Natanzon potentials in terms of Heun-type equations

These differential equations and their solutions are more general

The potentials contain [more parameters](#) and more (5 or 6) terms

They contain the (confluent) hypergeometric case, i.e. the Natanzon potentials

Problem: they are much less elaborated mathematically

Type	$Q(z)$	$R(z)$	$\exp\left(\frac{1}{2}\int^z Q(z)dz\right)$
Heun	$\frac{\gamma}{z-a_1} + \frac{\delta}{z-a_2} + \frac{\epsilon}{z-a_3}$	$\frac{\alpha\beta z-q}{(z-a_1)(z-a_2)(z-a_3)}$	$(z-a_1)^{\gamma/2}(z-a_2)^{\delta/2}(z-a_3)^{\epsilon/2}$
Confluent Heun	$4p + \frac{\gamma}{z} + \frac{\delta}{z-1}$	$\frac{4p\beta-\sigma}{(z-1)} + \frac{\sigma}{z}$	$z^{\gamma/2}(z-1)^{\delta/2}\exp(2pz)$
Bi-confluent Heun	$\frac{\gamma}{z} + \delta + \epsilon z$	$\alpha - \frac{q}{z}$	$z^{\gamma/2}\exp(\delta z/2 + \epsilon z^2/4)$
Doubly confluent Heun	$\frac{\delta}{z^2} + \frac{\gamma}{z} + 1$	$\frac{\alpha}{z} - \frac{q}{z^2}$	$z^{\gamma/2}\exp(-\delta/(2z) + z/2)$
Triple confluent Heun	$\gamma z + z^2$	$\alpha z - q$	$\exp(\gamma z^2/4 + z^3/6)$
Hypergeometric (H, CH)	$\frac{c}{z} + \frac{a+b+1-c}{z-1}$	$\frac{ab}{(z-1)} - \frac{ab}{z}$	$z^{c/2}(z-1)^{(a+b+1-c)/2}$
Confluent hypergeometric (CH, BCH, DCH)	$\frac{b}{z} - 1$	$-\frac{a}{z}$	$z^{b/2}\exp(-z/2)$

Apply the method to the confluent Heun equation

$$\begin{aligned}
 E - V(x) = & \frac{z'''(x)}{2z'(x)} - \frac{3}{4} \left(\frac{z''(x)}{z'(x)} \right)^2 \\
 & + (z'(x))^2 \left(-\frac{\alpha^2}{4} + \frac{2\sigma - \alpha\gamma + \delta\gamma}{2z(x)} + \frac{2\alpha\beta - 2\sigma - \alpha\delta - \gamma\delta}{2(z(x) - 1)} \right. \\
 & \left. + \frac{\gamma(2 - \gamma)}{4z^2(x)} + \frac{\delta(2 - \delta)}{4(z(x) - 1)^2} \right)
 \end{aligned}$$

Define z with

$$\left(\frac{dz}{dx} \right)^2 \Phi(z(x)) \equiv \left(\frac{dz}{dx} \right)^2 \frac{\phi(z(x))}{(z(x)(z(x) - 1))^2} = C ,$$

where

$$\phi(z(x)) = p_1 z^2(x)(z(x) - 1)^2 + p_2 z(x)(z(x) - 1)^2 + p_3 z^2(x)(z(x) - 1) + p_4 (z(x) - 1)^2 + p_5 z^2(x) .$$

Then the pre-factor of the solutions is

$$f(x) \sim \phi^{1/4} \exp \left(\frac{\alpha}{2} z(x) \right) (z(x))^{(\gamma-1)/2} (z(x) - 1)^{(\delta-1)/2}$$

The potential is

$$\begin{aligned}
 V(x) = & -\frac{z'''(x)}{2z'(x)} + \frac{3}{4} \left(\frac{z''(x)}{z'(x)} \right)^2 + \frac{C}{\phi(z(x))} \left[s_1 z^2(x)(z(x) - 1)^2 + s_2 z(x)(z(x) - 1)^2 \right. \\
 & \left. + s_3 z^2(x)(z(x) - 1) + s_4 (z(x) - 1)^2 + s_5 z^2(x) \right] .
 \end{aligned}$$

The parameters and E are related by

$$\begin{aligned}s_1 - p_1 \frac{E}{C} - \frac{\alpha^2}{4} &= 0 , \\s_2 - p_2 \frac{E}{C} + \sigma - \frac{\alpha\gamma}{2} + \frac{\delta\gamma}{2} &= 0 , \\s_3 - p_3 \frac{E}{C} + \alpha\beta - \sigma - \frac{\alpha\delta}{2} - \frac{\gamma\delta}{2} &= 0 , \\s_4 - p_4 \frac{E}{C} - \frac{\gamma}{2} \left(\frac{\gamma}{2} - 1 \right) &= 0 , \\s_5 - p_5 \frac{E}{C} - \frac{\delta}{2} \left(\frac{\delta}{2} - 1 \right) &= 0 .\end{aligned}$$

Now take the following substitutions:

$$p_2 = 1, \quad p_i = 0, \quad i \neq 2$$

$$\text{Then} \quad \phi(z) = z(z-1)^2 \quad \longrightarrow \quad z(r) = -\frac{C}{4}r^2$$

$$\text{Take also } \beta = -N \quad \gamma = \alpha + 1 \quad \delta = -2 \quad \sigma = (2 - N)\alpha$$

$$C = -2\omega/2 \quad \alpha = l + 1/2 \quad s_2 = 1$$

Then

$$V(x) = \frac{\omega^2}{4}x^4 + \frac{l(l+1)}{x^2} + \frac{4\omega}{\omega x^2 + 2l + 1} - \frac{8\omega(2l+1)}{(\omega x^2 + 2l + 1)^2} ,$$

$$E_N = \omega(2N + l - 1/2)$$

This is a **polynomial solution** of the confluent Heun equation.

The CHE reduces to that of the X_1 type exceptional Laguerre polynomials

$$\psi_N(x) \sim \frac{x^{l+1}}{\omega x^2 + 2l + 1} \exp\left(-\frac{\omega}{4}x^2\right) \hat{L}_N^{(l+1/2)}\left(\frac{\omega}{2}x^2\right) .$$

Potentials related to the X_1 type exceptional Jacobi polynomials also follow from the symmetric form of the CHE

See *GL Symmetry* **15** (2023) 461 for a systematic discussion of polynomial solutions of the CHE

Apply the method to the bi-confluent Heun equation

$$\begin{aligned}
 E - V(x) = & \frac{z'''(x)}{2z'(x)} - \frac{3}{4} \left(\frac{z''(x)}{z'(x)} \right)^2 \\
 & + (z'(x))^2 \left[-\frac{\gamma}{2} \left(\frac{\gamma}{2} - 1 \right) \frac{1}{z^2} - \left(q + \frac{\gamma\delta}{2} \right) \frac{1}{z} \right. \\
 & \left. + \left(\alpha - \frac{\epsilon}{2} - \frac{\delta^2}{4} - \frac{\gamma\epsilon}{2} \right) - \frac{\delta\epsilon}{2}z - \frac{\epsilon^2}{4}z^2 \right]
 \end{aligned}$$

Define z with

$$\left(\frac{dz}{dx} \right)^2 \Phi(z(x)) = C ,$$

where

$$\Phi(z(x)) = p_1 \frac{1}{z^2(x)} + p_2 \frac{1}{z(x)} + p_3 + p_4 z(x) + p_5 z^2(x) .$$

Then the pre-factor of the solutions is

$$f(x) \sim \Phi^{1/4}(z(x))(z(x))^{\gamma/2} \exp \left(\frac{\delta}{2}z(x) + \frac{\epsilon}{4}z^2 \right)$$

The potential is

$$V(x) = -\frac{z'''(x)}{2z'(x)} + \frac{3}{4} \left(\frac{z''(x)}{z'(x)} \right)^2 + \frac{C}{\Phi(z(x))} \left[s_1 \frac{1}{z^2(x)} + s_2 \frac{1}{z(x)} + s_3 + s_4 z(x) + s_5 z^2(x) \right] .$$

The parameters and E are related by

$$\begin{aligned} s_1 - p_1 \frac{E}{C} - \frac{\gamma}{2} \left(\frac{\gamma}{2} - 1 \right) &= 0 , \\ s_2 - p_2 \frac{E}{C} - \left(q + \frac{\gamma\delta}{2} \right) &= 0 , \\ s_3 - p_3 \frac{E}{C} + \left(\alpha - \frac{\epsilon}{2} - \frac{\delta^2}{4} - \frac{\gamma\epsilon}{2} \right) &= 0 , \\ s_4 - p_4 \frac{E}{C} - \frac{\delta\epsilon}{2} &= 0 , \\ s_5 - p_5 \frac{E}{C} - \frac{\epsilon^2}{4} &= 0 . \end{aligned}$$

Now take the following substitutions:

$$p_2 = 1, \quad p_i = 0, \quad i \neq 2$$

Then $\Phi(z) = 1/z \longrightarrow z(r) = -\frac{C}{4}x^2$

Take also $\gamma = 2s \quad \delta = -4b/C \quad \epsilon = -16a/C^2 \quad \alpha = 16aM \quad C = 4$

Then

$$V(x) = \frac{(2s - 1/2)(2s - 3/2)}{x^2} + \left(b^2 - 4a(s + M + 1/2) \right) r^2 + 2abr^4 + a^2x^2$$

$$E = 4bs - 4q$$

Polynomial solution of the bi-confluent Heun equation.

Ishkhanyan, Lévai Phys. Scr. 95 (2020) 085202

Apply the method to the doubly confluent Heun equation

$$\begin{aligned}
 E - V(x) = & \frac{z'''(x)}{2z'(x)} - \frac{3}{4} \left(\frac{z''(x)}{z'(x)} \right)^2 \\
 & + (z'(x))^2 \left[-\frac{\delta^2}{4} \frac{1}{z^4} - \left(\delta - \frac{\gamma\delta}{2} \right) \frac{1}{z^3} \right. \\
 & \left. + \left(-q + \frac{\gamma}{2} - \frac{\gamma^2}{4} - \frac{\delta}{2} \right) \frac{1}{z^2} + \left(\alpha - \frac{\gamma}{2} \right) \frac{1}{z} - \frac{1}{4} \right]
 \end{aligned}$$

Define z with

$$\left(\frac{dz}{dx} \right)^2 \Phi(z(x)) = C ,$$

where

$$\Phi(z(x)) = p_1 \frac{1}{z^4(x)} + p_2 \frac{1}{z^3(x)} + p_3 \frac{1}{z^2(x)} + p_4 \frac{1}{z(x)} + p_5 .$$

Then the pre-factor of the solutions is

$$f(x) \sim \Phi^{1/4}(z(x)) (z(x))^{\gamma/2} \exp(-\delta/(2z(x)) + z(x)/2)$$

The potential is

$$V(x) = -\frac{z'''(x)}{2z'(x)} + \frac{3}{4} \left(\frac{z''(x)}{z'(x)} \right)^2 + \frac{C}{\Phi(z(x))} \left[s_1 \frac{1}{z^4(x)} + s_2 \frac{1}{z^3(x)} + s_3 \frac{1}{z^2(x)} + s_4 \frac{1}{z(x)} + s_5 \right] .$$

The parameters and E are related by

$$\begin{aligned}s_1 - p_1 \frac{E}{C} - \frac{\delta^2}{4} &= 0 , \\s_2 - p_2 \frac{E}{C} + \delta - \frac{\gamma\delta}{2} &= 0 , \\s_3 - p_3 \frac{E}{C} - q + \frac{\gamma}{2} - \frac{\gamma^2}{4} - \frac{\delta}{2} &= 0 , \\s_4 - p_4 \frac{E}{C} + \alpha - \frac{\gamma}{2} &= 0 , \\s_5 - p_5 \frac{E}{C} - \frac{1}{4} &= 0 .\end{aligned}$$

The problem of solving a differential equation is traded for solving a set of algebraic equations

Perhaps things would be simpler for **polynomial solutions**

Apply the method to the triple confluent Heun equation

$$E - V(x) = \frac{z'''(x)}{2z'(x)} - \frac{3}{4} \left(\frac{z''(x)}{z'(x)} \right)^2 + (z'(x))^2 \left[-\frac{1}{4}z^4 - \frac{\gamma}{2}z^3 - \frac{\gamma^2}{4}z^2(\alpha - 1)z - (q + \frac{\gamma}{2}) \right]$$

Define z with

$$\left(\frac{dz}{dx} \right)^2 \Phi(z(x)) = C ,$$

where

$$\Phi(z(x)) = p_1 z^4(x) + p_2 z^3(x) + p_3 z^2(x) + p_4 z(x) + p_5 .$$

Then the pre-factor of the solutions is

$$f(x) \sim \Phi^{1/4}(z(x)) \exp \left(\gamma z^2(x)/4 + z^3(x)/6 \right)$$

The potential is

$$V(x) = -\frac{z'''(x)}{2z'(x)} + \frac{3}{4} \left(\frac{z''(x)}{z'(x)} \right)^2 + \frac{C}{\Phi(z(x))} \left[s_1 z^4(x) + s_2 z^3(x) + s_3 z^2(x) + s_4 z(x) + s_5 \right] .$$

The parameters and E are related by

$$\begin{aligned}s_1 - p_1 \frac{E}{C} - \frac{1}{4} &= 0 , \\s_2 - p_2 \frac{E}{C} - \frac{\gamma}{2} &= 0 , \\s_3 - p_3 \frac{E}{C} - q + \frac{\gamma}{2} - \frac{\gamma^2}{4} - \frac{\delta}{2} &= 0 , \\s_4 - p_4 \frac{E}{C} - \frac{\gamma^2}{4} &= 0 , \\s_5 - p_5 \frac{E}{C} - (q + \frac{\gamma}{2}) &= 0 .\end{aligned}$$

Again:

The problem of solving a differential equation is traded for solving a set of algebraic equations

Perhaps things would be simpler for **polynomial solutions**

Discussion

Variable transformations and SUSYQM help a lot to find solvable potentials

They also help in the classification of solvable potentials

Natanzon-class potentials: $F(z)$ is the (confluent) hypergeometric function

- Bound states are described by classical orthogonal polynomials

Potentials beyond the Natanzon class?

- They have been found using different methods

QES

Exceptional orthogonal polynomials

They can be discussed in a unified form in terms of Heun-type equations

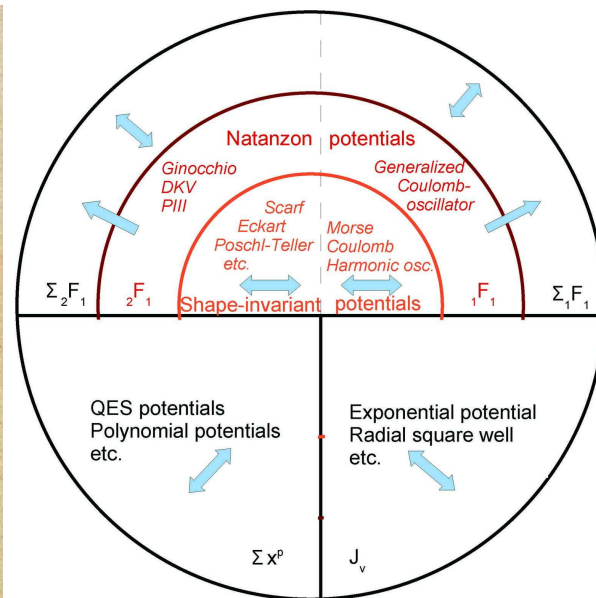
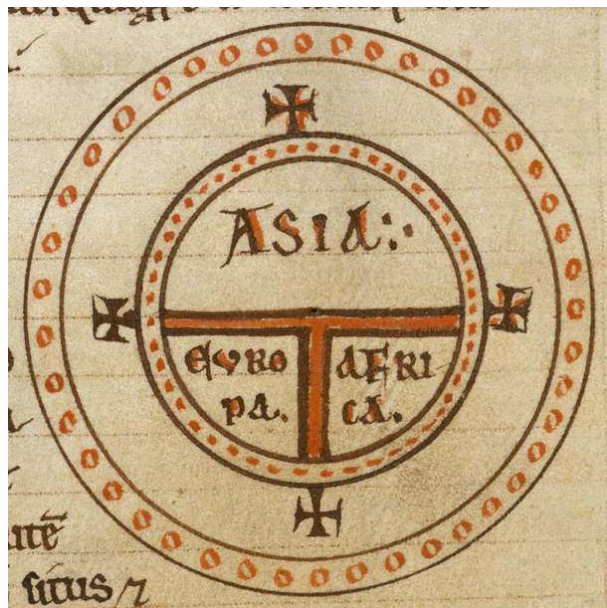
- Polynomial solutions of the CHE: rationally extended harmonic oscillator
- Polynomial solutions of the CHE: rationally extended Scarf II potential (not shown)
- Polynomial solutions of the BHE: sextic oscillator
- What about polynomial solutions of the DCH and TCH?

The solutions arise as the combination of two ordinary polynomials (recursion!)

This paves the way to SUSYQM: polynomial + its derivative

Natanzon-class potentials included as special cases

The world map of solvable potentials revisited



We were looking for islands, but found the Heun Ocean