# Analytic and Algebraic Methods in Physics XXI

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# Some recent applications of the Generalized Uncertainty Principle


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1. Introduction & Motivation

2. Quantum Correction to the Coulomb Potential in GUP Approach

3. Dirac equation with generalized gravitational interaction

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- A modified Schrödinger equation from a generalized uncertainty principle (GUP), with a quantum mechanically corrected gravitational interaction. The resulting equation cannot be solved by common exact approaches $\Rightarrow$ a Bethe-ansatz approach.

- Dirac equation with a generalized gravitational interaction which includes post-Newtonian (relativistic) and quantum corrections to the classical potential.
  The Bethe-ansatz approach is also proposed to attack this challenging problem.

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Obviously, those are not well studied problems in mathematical physics, where the differential equations that appear, whether relativistic or non-relativistic, are of the first or second order.

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(d) **Ground and first excited states** are explicitly determined.

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with an effective potential

$$V_e(x) := \frac{2m}{\hbar^2}\left(V(x) - E_n^G\right) + \left(\frac{2m}{\hbar^2}\right)^2 \beta \left(V(x) - E_n\right)^2,$$

with $E_n$ the $\beta = 0$ eigenvalues of the energy and $E_n^G$ the eigenenergies for $\beta \neq 0$.

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$$\frac{2m}{\hbar^2} V(x) = \frac{\alpha_1}{x} + \frac{\alpha_2}{x^2} + \frac{\alpha_3}{x^3} + \frac{\alpha_4^2}{x^4}, \quad \alpha_4 > 0,\ \alpha_1 < 0,$$

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$$V_e(x) = \gamma_0 + \frac{\gamma_1}{x} + \frac{\gamma_2}{x^2} + \frac{\gamma_3}{x^3} + \frac{\gamma_4}{x^4} + \frac{\gamma_5}{x^5} + \frac{\gamma_6}{x^6} + \frac{\gamma_7}{x^7} + \frac{\gamma_8}{x^8}\,,$$

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in which the $\gamma_i$, $i = 0, 1, \ldots 8$ are as follows

$$\gamma_0 = \beta\epsilon_n^2 - \epsilon_n^G, \qquad \gamma_1 = \alpha_1\big(1 - 2\beta\epsilon_n\big),$$

$$\gamma_2 = \alpha_2 + \beta\big(\alpha_1^2 - 2\alpha_2\epsilon_n\big), \qquad \gamma_3 = \alpha_3 + 2\beta\left(\alpha_1\alpha_2 - \alpha_3\epsilon_n\right),$$

$$\gamma_4 = \alpha_4^2 + \beta\big(2\alpha_1\alpha_3 + \alpha_2^2 - 2\alpha_4^2\epsilon_n\big), \qquad \gamma_5 = 2\beta(\alpha_1\alpha_4^2 + \alpha_2\alpha_3),$$

$$\gamma_6 = \beta\left(2\alpha_2\alpha_4^2 + \alpha_3^2\right), \qquad \gamma_7 = 2\beta\alpha_3\alpha_4^2, \qquad \gamma_8 = \beta\alpha_4^4,$$

where we have introduced the notation

$$\epsilon_n = \frac{2m}{\hbar^2} E_n, \qquad \epsilon_n^G = \frac{2m}{\hbar^2} E_n^G.$$

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Let us start the analysis of the ordinary case ($\beta = 0$), as the solutions of the model with GUP depend on them.

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**The ordinary case $\beta = 0$ (not GUP)** has already been studied via the ansatz method. Here, we will obtain the general solutions of the model using the Lie-algebraic method within the framework of quasi-exact solvability.

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The Schrödinger equation with Coulomb–4 potential appears in the form

$$\left(-\frac{d^2}{dx^2} + \frac{\alpha_1}{x} + \frac{\alpha_2}{x^2} + \frac{\alpha_3}{x^3} + \frac{\alpha_4^2}{x^4} - \epsilon_n\right)\psi_n(x) = 0.$$

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Due to the asymptotic behavior of the wave function $\psi_n(x)$, we use the ansatz

$$\psi_n(x) = x^\delta \exp\left[-\left(x\sqrt{-\epsilon_n} + \frac{\alpha_4}{x}\right)\right]\varphi_n(x), \qquad \delta = 1 + \frac{\alpha_3}{2\alpha_4} > 0,$$

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which transforms Schrödinger equation (9) into the form

$$\left\{-x^2\frac{d^2}{dx^2} - 2\left(\alpha_4 + \delta\, x - \sqrt{-\epsilon_n}\, x^2\right)\frac{d}{dx} + (\lambda_1\, x + \lambda_2)\right\}\varphi_n(x) = 0,$$

where

$$\lambda_1 = \alpha_1 + \left(2 + \frac{\alpha_3}{\alpha_4}\right)\sqrt{-\epsilon_n}, \qquad \lambda_2 = \alpha_2 - \frac{\alpha_3^2}{4\alpha_4^2} - \frac{\alpha_3}{2\alpha_4} + 2\alpha_4\sqrt{-\epsilon_n}.$$

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**The Lie-algebraic approach:** Following the standard idea of quasi-exact solvability, we find that if the constraint

$$\lambda_1 = -2n\sqrt{-\epsilon_n}$$

holds, the Schrödinger equation can then be expressed as a quasi-exactly solvable (QES) differential operator in the Lie-algebraic form

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with

$$H_{qes} = -\mathcal{J}_n^+\mathcal{J}_n^- + 2\sqrt{-\epsilon_n}\,\mathcal{J}_n^+ - 2\alpha_4\,\mathcal{J}_n^- - \Big(2\delta + n\Big)\mathcal{J}_n^0 - \frac{n^2}{2} - n\delta + \lambda_2\,.$$

Here,

$$\mathcal{J}_n^+ = x^2\,\frac{d}{dx} - n\,x, \qquad \mathcal{J}_n^0 = x\,\frac{d}{dx} - \frac{n}{2}, \qquad \mathcal{J}_n^- = \frac{d}{dx},$$

are the generators of the $sl(2)$ Lie algebra satisfying the commutation relations

$$[\mathcal{J}_n^+, \mathcal{J}_n^-] = -2\mathcal{J}_n^0, \qquad [\mathcal{J}_n^\pm, \mathcal{J}_n^0] = \mp\mathcal{J}_n^\pm.$$

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These operators leave invariant the $(n+1)$-dimensional linear space of polynomials

$$\varphi_n(x) = \sum_{k=0}^{n} c_k \, x^k,$$

where the coefficients $c_k$ satisfy the three-term recursion relation $(c_{-1} = c_{n+1} = 0)$

$$c_{k+1} = \frac{(\lambda_2 - 2k\,\delta - k(k-1))\,c_k - 4\sqrt{-\epsilon_n}\,c_{k-1}}{2(k+1)\,\alpha_4}, \qquad k = 0, 1, \ldots, n,$$

assuming $c_{-1} = 0$ and $c_{n+1} = 0$.

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assuming $c_{-1} = 0$ and $c_{n+1} = 0$. Equivalently, the recursion relation (11) can be rewritten as a tridiagonal matrix equation the nontrivial solutions of which are

$$\begin{vmatrix} \lambda_2 & -2\alpha_4 & & & \\ -2n\sqrt{-\epsilon_n} & (\lambda_2 - 2\delta) & -4\alpha_4 & & \\ & -2(n-1)\sqrt{-\epsilon_n} & \ddots & \ddots & \\ & & \ddots & \ddots & -2n\,\alpha_4 \\ & & & -2\sqrt{-\epsilon_n} & (n - n^2 + \lambda_2 - 2n\,\delta) \end{vmatrix} = 0.$$

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On the other hand, from the expressions of $\lambda_1$ and $\lambda_2$ the following expression of the energy in closed form can be obtained

$$\epsilon_n = -\frac{\alpha_1^2\,\alpha_4^2}{(\alpha_3 + 2(n+1)\alpha_4)^2},$$

provided $\alpha_1 < 0$.

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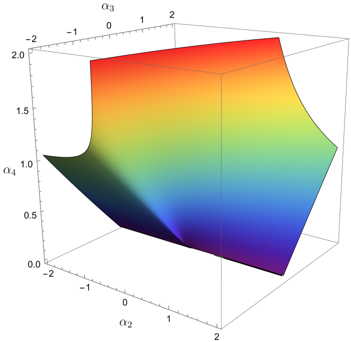
For the sake of clarity, next we will find the explicit solutions of the ground state and the first excited state.

## Ground state and the associated wave function

$$\epsilon_0 = -\frac{\alpha_1^2\,\alpha_4^2}{(\alpha_3 + 2\alpha_4)^2}, \qquad \psi_0(x) = c_0\; x^{1+\alpha_3/(2\alpha_4)}\; \exp\left[-\left(x\sqrt{-\epsilon_0} + \frac{\alpha_4}{x}\right)\right],$$

where the restriction on the parameters of the potential is determined by

$$4\alpha_2\,\alpha_4^2 - \alpha_3^2 - 2\alpha_3\,\alpha_4 + 8\alpha_4^3\sqrt{-\epsilon_0} = 0.$$




$\alpha_1 = -1/10$

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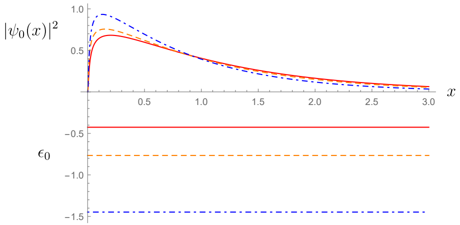
On the left, the wave function of the ground state, together with the corresponding energy,



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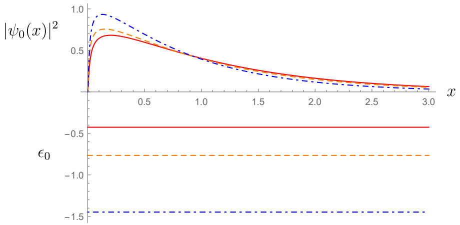


On the right, the form of the potential.

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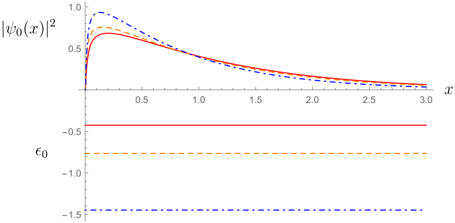
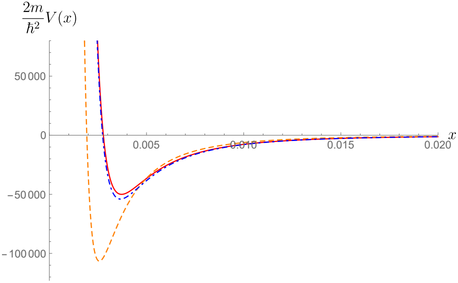



On the right, the form of the potential.



**First excited state ($n = 1$) and the associated wave function**

The energy and the corresponding wave function are given by

$$\epsilon_1 = -\frac{\alpha_1^2\,\alpha_4^2}{(\alpha_3 + 4\alpha_4)^2}\,, \qquad \psi_1(x) = (c_0 + c_1 x)\, x^{1+\alpha_3/(2\alpha_4)}\, \exp\left[-\left(x\sqrt{-\epsilon_1} + \frac{\alpha_4}{x}\right)\right].$$

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The restriction on the parameters of the potential is given by

$$64\alpha_1^2\alpha_4^8 + 16\alpha_4^4(\alpha_3 + 4\alpha_4)(\alpha_3^2 + 4\alpha_3\alpha_4 + 8\alpha_4^2 - 4\alpha_2\alpha_4^2)\alpha_1$$
$$+(\alpha_3 + 4\alpha_4)^2(\alpha_3^2 + 2\alpha_3\alpha_4 - 4\alpha_2\alpha_4^2)(\alpha_3^2 + 6\alpha_3\alpha_4 + 8\alpha_4^2 - 4\alpha_2\alpha_4^2) = 0.$$

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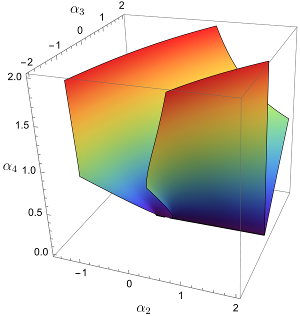


$$\alpha_1 = -\frac{1}{5}$$

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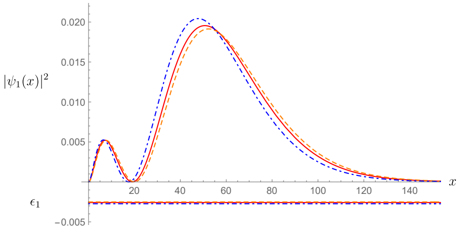
On the left, the wave function of the first excited state, together with the corresponding energy,



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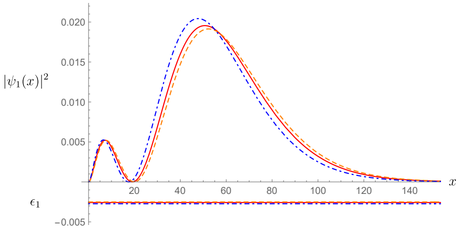


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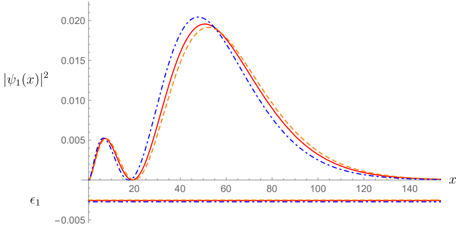


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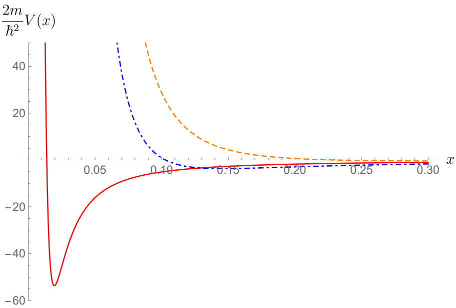

# 2.– Quantum Correction to the Coulomb Potential in GUP

**The ordinary case as a double-confluent Heun equation:**

# 2.– Quantum Correction to the Coulomb Potential in GUP

**The ordinary case as a double-confluent Heun equation:** By changing the independent variable $y = 2\sqrt{-\epsilon_n}\, x$, the Schrödinger differential equation

$$\left\{-x^2 \frac{d^2}{dx^2} - 2\left(\alpha_4 + \delta\, x - \sqrt{-\epsilon_n}\, x^2\right)\frac{d}{dx} + (\lambda_1\, x + \lambda_2)\right\}\varphi_n(x) = 0,$$

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is transformed into the double-confluent Heun equation

$$\left\{y^2 \frac{d^2}{dy^2} + \left(-y^2 + \rho\, y + \eta\right) \frac{d}{dy} - (\omega\, y + \lambda_2)\right\} \varphi_n(y) = 0,$$

in which we denote $\rho = 2 + \frac{\alpha_3}{\alpha_4}$, $\eta = 4\alpha_4\sqrt{-\epsilon_n}$, $\omega = 1 + \frac{\alpha_1}{2\sqrt{-\epsilon_n}} + \frac{\alpha_3}{2\alpha_4}$.

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Its regular solutions at origin are given by $(h_0 = 1)$

$$\varphi := \varphi_n(y\,;\rho,\eta,\omega,\lambda_2) = \sum_{n=0}^{\infty} h_n\,x^n,$$

$$h_{n+2} = \frac{(\lambda_2 - n(n+\rho+1) - \rho)\,h_{n+1} + (n+\omega)\,h_n}{(n+2)\eta} \qquad \text{and} \qquad h_1 = \frac{\lambda_2}{\eta}\,h_0\,,$$

Consequently, $\varphi$ can admit polynomial solution of degree $m$ if $(m+\omega)$ and $h_{m+1}$ vanish simultaneously. In this way, general solutions to the problem can be obtained in terms of the associated Heun functions.

# 2.– Quantum Correction to the Coulomb Potential in GUP

**The GUP case, $\beta \neq 0$.**

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Having obtained the solutions of the Coulomb–4 model without GUP in the previous section, let us now return to the modified Schrödinger equation in a formalism with GUP, that is, to the equations

$$\left[-\frac{d^2}{dx^2} + V_e(x)\right] \psi_n^G(x) = 0, \; V_e(x) := \frac{2m}{\hbar^2} \left(V(x) - E_n^G\right) + \left(\frac{2m}{\hbar^2}\right)^2 \beta \left(V(x) - E_n\right)^2 ,$$

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To ensure proper asymptotic behavior of the wave function $\psi_n^G(x)$, after inspecting the differential equation we propose the following ansatz

$$\psi_n^G(x) = x^f\, e^{g(x)}\, \varphi_n^G(x), \qquad g(x) = -a\,x - \frac{b}{x} - \frac{c}{x^2} - \frac{d}{x^3},$$

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It can be seen that the previous parameters are determined by:

# 2.– Quantum Correction to the Coulomb Potential in GUP

$$a = \sqrt{\gamma_0} = \sqrt{\beta\epsilon_n^2 - \epsilon_n^G} > 0, \qquad b = -\frac{\gamma_7^2 - 4\gamma_6\,\gamma_8}{8\,(\gamma_8)^{3/2}} = \alpha_2\,\sqrt{\beta},$$

$$c = \frac{\gamma_7}{4\sqrt{\gamma_8}} = \frac{1}{2}\,\alpha_3\,\sqrt{\beta}, \qquad d = \frac{\sqrt{\gamma_8}}{3} = \frac{1}{3}\alpha_4^2\,\sqrt{\beta},$$

$$f = 2 + \frac{8\gamma_5\,\gamma_8^2 - 4\gamma_6\,\gamma_7\,\gamma_8 + \gamma_7^3}{16\,(\gamma_8)^{5/2}} = 2 + \alpha_1\sqrt{\beta} > 0\,.$$

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Then, the differential equation for $\varphi_n^G(x)$ simplifies to

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where

$$\begin{aligned} P_4(x) = &\; x^4, \\ Q_4(x) = &\; 6d + 4c\,x + 2b\,x^2 + 2f\,x^3 - 2a\,x^4, \\ W_3(x) = &\; (b^2 - 6ad + 4cf - 6c - \gamma_4) + (2(bf - b - 2ac) - \gamma_3)x \\ &+ (f(f-1) - 2ab - \gamma_2)x^2 - (\gamma_1 + 2af)x^3. \end{aligned}$$

To find the solution $\varphi_n^G(x)$ we use the general Bethe ansatz method, looking for polynomial solutions of the form

$$\varphi_n^G(x) = \begin{cases} 1, & n = 0, \\ \displaystyle\prod_{i=1}^{n}(x - x_i), & n \in \mathbb{N}, \end{cases}$$

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After some calculations we get the energy relation: for a given $n$, the energy $\epsilon_n^G$ is

$$\epsilon_n^G(\epsilon_n; \alpha_1, \beta) = -\frac{\alpha_1^2}{4}\left(\frac{2\beta\,\epsilon_n - 1}{\alpha_1\sqrt{\beta} + n + 2}\right)^2 + \beta\,\epsilon_n^2\,.$$

The denominator of the fraction, $\alpha_1\sqrt{\beta} + n + 2$, is always positive.

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The denominator of the fraction, $\alpha_1\sqrt{\beta} + n + 2$, is always positive.

As in the ordinary case (without a GUP, $\beta = 0$), we will now look for explicit solutions for the ground state and the first excited state.

## Ground state and the associated wave function

For $n = 0$, the energy of ground state, $\epsilon_0^G$, is determined in closed form:

$$\epsilon_0^G(\epsilon_0; \alpha_1, \beta) = -\frac{\alpha_1^2}{4}\left(\frac{2\beta\,\epsilon_0 - 1}{\alpha_1\sqrt{\beta} + 2}\right)^2 + \beta\,\epsilon_0^2\,.$$

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The explicit form of the associated wave function is given by

$$\psi_0^G(x) = C_0\;x^{2+\alpha_1\sqrt{\beta}}\;\exp\left[-x\sqrt{\beta\,\epsilon_0^2 - \epsilon_0^G} - \frac{\sqrt{\beta}}{2}\left(\frac{2\alpha_4^2}{3x^3} + \frac{\alpha_3}{x^2} + \frac{2\alpha_2}{x}\right)\right],$$

where $C_0$ is the normalization constant and the parameters of the potential satisfy some constraints.

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For $n = 1$, the first excited state energy, $\epsilon_1^G$, is given by

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**Roadmap:**

(A) A review of the essential formulae of the Dirac equation in the desired metric.

(B) Using the Bethe-ansatz approach, we report the general solution for arbitrary $n$, in particular solutions for the ground and the first excited states.

## (A) Curved spacetime Dirac equation with a generalized metric

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The Dirac equation $(i\gamma^{\mu}\nabla_{\mu} - mc)\,\Psi = 0$ will be considered for the metric

$$g_{\mu\nu} = \mathrm{diag}\left(e^{2f(r)}, -e^{2g(r)}, -r^2, -r^2\sin^2\theta\right),$$

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In curved spacetime $\nabla_{\mu} = \partial_{\mu} + iA_{\mu}/c + \Omega_{\mu}$, with $\partial_{\mu}$ the covariant derivative on flat spacetime and $\Omega_{\mu}$ the spin connection. Choosing $A_{\mu} = (V(r), cA_r(r), 0, 0)$, the spinor wave function is

$$\Psi_c(r,\theta,\phi) = N\begin{pmatrix} R_1(r)\,\mathcal{Y}^{|m|\,j}_{j+1/2}(\theta,\phi) \\ iR_2(r)\,\mathcal{Y}^{|m|\,j}_{j-1/2}(\theta,\phi)\end{pmatrix}.$$

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$$\mathcal{Y}_l^{j=l\pm 1/2\; m}(\theta,\phi) = \frac{1}{\sqrt{2l+1}} \begin{pmatrix} \pm\sqrt{l \pm m + \frac{1}{2}}\ \ Y_l^{m-1/2}(\theta,\phi) \\ \sqrt{l \mp m + \frac{1}{2}}\ \ Y_l^{m+1/2}(\theta,\phi) \end{pmatrix}$$

are the spinor spherical harmonics, being $Y_l^m(\theta,\phi)$ the usual spherical harmonics.

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We manipulate Dirac equation $\left(i\gamma^\mu \nabla_\mu - mc\right)\Psi = 0$ with the assumptions:

- $m = \hbar = 1$, $c = 1/\alpha$, with $\alpha$ being the fine structure constant.
- $f(r) = g(r)$, with $e^{f(r)} = 1 + \alpha^2 U(r)$,
- $V(r) = a\, z(r)$ and $U(r) = b\, z(r)$,
- $R_1(r) = \frac{u(r)}{r} e^{-f(r)/2}$ and $R_2(r) = \frac{v(r)}{r} e^{-f(r)/2}$,
- $\begin{pmatrix} \rho_1(r) \\ \rho_2(r) \end{pmatrix} = \mathbf{U}' \begin{pmatrix} u(r) \\ v(r) \end{pmatrix}, \quad \mathbf{U}' = \exp(i\eta\sigma_2),$
- $A_r(r) = \frac{\alpha C}{S}\left[\frac{V(r)}{C} - U(r)\right] - \frac{\lambda}{r}\left[1 + \alpha^2 U(r)\right], \;\; C = \cos 2\eta, S = \sin 2\eta,$

# 3.– Dirac eqn. with generalized gravitational interaction

We manipulate Dirac equation $(i\gamma^\mu \nabla_\mu - mc)\,\Psi = 0$ with the assumptions:

- $m = \hbar = 1$, $c = 1/\alpha$, with $\alpha$ being the fine structure constant.
- $f(r) = g(r)$, with $e^{f(r)} = 1 + \alpha^2 U(r)$,
- $V(r) = a\,z(r)$ and $U(r) = b\,z(r)$,
- $R_1(r) = \frac{u(r)}{r} e^{-f(r)/2}$ and $R_2(r) = \frac{v(r)}{r} e^{-f(r)/2}$,
- $\begin{pmatrix} \rho_1(r) \\ \rho_2(r) \end{pmatrix} = \mathbf{U}' \begin{pmatrix} u(r) \\ v(r) \end{pmatrix}$, $\quad \mathbf{U}' = \exp(i\eta\sigma_2)$,
- $A_r(r) = \frac{\alpha C}{S} \left[ \frac{V(r)}{C} - U(r) \right] - \frac{\lambda}{r} \left[ 1 + \alpha^2 U(r) \right]$, $C = \cos 2\eta, S = \sin 2\eta$,

and we obtain the final equation for the component $\rho_1(r)$

$$\boxed{\left[ \frac{d^2}{dr^2} + \frac{\alpha}{S}(aC - b)z'(r) - 2(b + \epsilon a)z(r) - \frac{\alpha^2}{S^2}(aC - b)^2 z^2(r) + \frac{\epsilon^2 - 1}{\alpha^2} \right] \rho_1(r) = 0}$$

# 3.– Dirac eqn. with generalized gravitational interaction

## (B) Solutions by the Bethe-ansatz approach

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Let us consider the *potential* $z(r)$ as in the previous Schrödinger-like equation

$$z(r) = \frac{u}{r} + \frac{v}{r^2} + \frac{w}{r^3}, \qquad u, v, w < 0,$$

that has been investigated before in the GUP framework.

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$$\left[\frac{d^2}{dr^2} + \frac{\epsilon_n^2 - 1}{\alpha^2} - \frac{2u(a\epsilon_n + b)}{r} + \frac{1}{S^2}\left(\frac{\Lambda_2}{r^2} + \frac{\Lambda_3}{r^3} + \frac{\Lambda_4}{r^4}\right) - \frac{\alpha^2(b - aC)^2}{S^2}\left(\frac{w^2}{r^6} + \frac{2vw}{r^5}\right)\right]\rho_{1,n}(r) = 0,$$

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where $\Lambda_2$, $\Lambda_3$ and $\Lambda_4$ are some functions of $u, v, w, a, b, \alpha, S, C, \epsilon_n$. We have added the index $n$ to $\epsilon$ and $\rho_1(r)$ to distinguish some states from others.

# 3.– Dirac eqn. with generalized gravitational interaction

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Now, we propose

$$\rho_{1,n}(r) = e^{\Delta(r)}\,\mathcal{R}_{1,n}(r), \qquad \Delta(r) = \delta \ln r + \frac{\gamma}{r^2} + \frac{\beta}{r} + \lambda r,$$

where $\mathcal{R}_{1,n}(r)$ is a polynomial, and $\lambda\,, \beta\,, \gamma < 0$, $\delta > 0$ are parameters to be find.

Consequently, the differential equation for $\mathcal{R}_{1,n}(r)$ simplifies to

$$\left\{ r^3 \frac{d^2}{dr^2} + \left(2\lambda r^3 + 2\delta r^2 - 2\beta r - 4\gamma\right) \frac{d}{dr} + \left(\xi_2\, r^2 + \frac{\xi_1}{S^2}\, r + \frac{\xi_0}{S^2}\right) \right\} \mathcal{R}_{1,n}(r) = 0,$$

where $\xi_0$, $\xi_1$, $\xi_2$ are known functions of $u, v, w, a, b, \alpha, S, C, \epsilon_n$.

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where $\xi_0$, $\xi_1$, $\xi_2$ are known functions of $u, v, w, a, b, \alpha, S, C, \epsilon_n$.
To solve this equation, we assume $\mathcal{R}_{1,n}(r)$ to be a polynomial of the form

$$\mathcal{R}_{1,n}(r) = \begin{cases} 1, & n = 0, \\ \prod_{i=1}^{n}(r - r_i), & n \in \mathbb{N}, \end{cases}$$

where $r_i$ are distinct roots to be determined. The general solutions are given by

$$\begin{aligned} &\xi_2 + 2n\lambda = 0, \\ &\xi_1 + 2\lambda \sum_{i=1}^{n} r_i + n(n-1) + 2n\delta = 0, \\ &\xi_0 + 2\lambda \sum_{i=1}^{n} r_i^2 + 2(\delta + n - 1) \sum_{i=1}^{n} r_i - 2n\beta = 0, \end{aligned}$$

the $r_i$ given by the Bethe ansatz eqns $\displaystyle\sum_{j=1,\ j\neq i}^{n} \frac{1}{r_i - r_j} + \frac{\lambda\, r_i^3 + \delta\, r_i^2 - \beta\, r_i - 2\gamma}{r_i^3} = 0.$

**Ground state and the associated wave function**

## Ground state and the associated wave function

For $n = 0$ it follows that the ground state energy $\epsilon_0$ is given by

$$\epsilon_0 = \frac{-4a\alpha^2\mathcal{A}^2bu^2 \pm \sqrt{4\alpha^2\mathcal{A}^2u^2(a^2-b^2)(\mathcal{B}-3\mathcal{A})^2 + (\mathcal{B}-3\mathcal{A})^4}}{4a^2\alpha^2\mathcal{A}^2u^2 + (\mathcal{B}-3\mathcal{A})^2},$$

with

$$\mathcal{A} := \sqrt{\alpha^2S^2(b-aC)^2}, \qquad \mathcal{B} := \alpha(b-aC)\left(2\alpha u(b-aC) - 3S\right).$$

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$$\mathcal{A} := \sqrt{\alpha^2S^2(b-aC)^2}, \qquad \mathcal{B} := \alpha(b-aC)\,(2\alpha u(b-aC)-3S)\,.$$

The associated wave function is $\rho_{1,0}(r) \propto e^{\Delta(r)}$, with

$$\Delta(r) = \left(\frac{3}{2}+\frac{\alpha w(b-aC)(2\alpha u(b-aC)-3S)}{2\sigma S^2}\right)\ln r - \frac{\sigma}{r^2}\left(\frac{rv}{w}+\frac{1}{2}\right) - \sqrt{\frac{1-\epsilon_0^2}{\alpha^2}}\,r,$$

The potential parameters $v$ and $w$ are given in terms of $u, a, b, C, S, \alpha, \epsilon_0$.

**First excited state and the associated wave function**

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For $n = 1$, the first excited-state energy, $\epsilon_1$, is given by

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The associated wave function is $\rho_{1,1}(r) \propto (r-r_1)\,e^{\Delta(r)}$, with

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The potential parameters $v$ and $w$ are given in terms of the Coulombic coefficient $u$ and $a, b, C, S, \alpha, \epsilon_0$.

The special case of the Coulomb interaction $v = w = 0$ can be obtained and compared the existing literature.

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**You are welcome to participate!** **See you next January in Valladolid!**