Renormalization of Point Scatterers in 2D & 3D, the Coincidence-Limit Problem and Dynamical Formulation of stationary scattering

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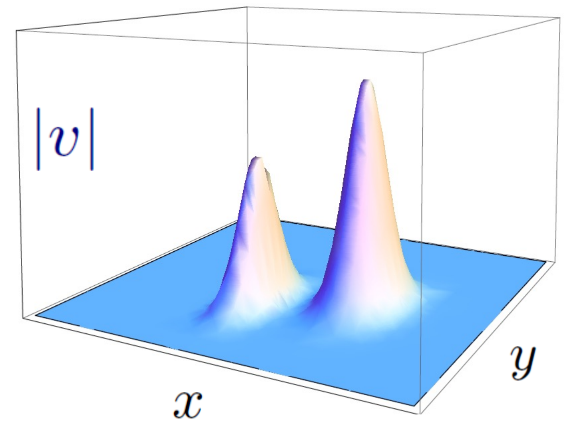
Outline:

- N^{th}-order Born approximation
- Multi-delta-function potentials in 2D & 3D, their renormalization, and coincidence-limit problem
- Dynamical formulation of stationary scattering
- Applications:
  - Consistent & finite treatment of multi-delta-function potentials in 2D & 3D
  - Potentials for which N^{th}-order Born approximation is exact.
- Conclusions

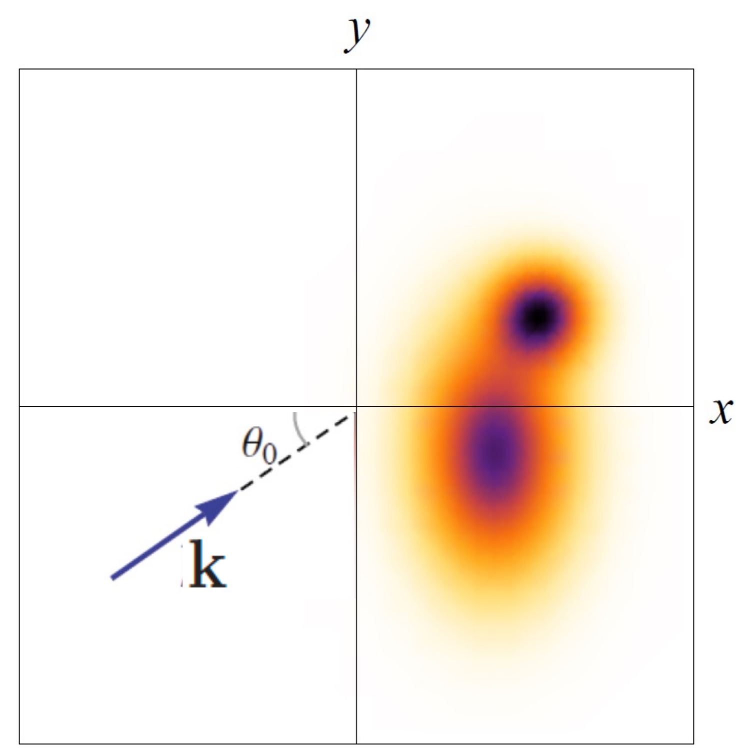
Scattering in 2D

$$\left[-\partial_x^2 - \partial_y^2 + v(x,y)\right]\psi(x,y) = k^2\psi(x,y)$$



$\mathbf{k} = k(\cos\theta_0, \sin\theta_0)$: Incident wavevector

θ_0: Incidence angle



Scattering in 2D

$$\left[-\partial_x^2 - \partial_y^2 + v(x,y)\right]\psi(x,y) = k^2\psi(x,y)$$

Lippmann-Schwinger eq./Green's fn. method:

$$\left.\begin{array}{c}(\widehat{\mathbf{p}}^2+\widehat{v})|\psi\rangle = k^2|\psi\rangle \\ \widehat{\mathbf{p}}^2|\mathbf{k}\rangle = k^2|\mathbf{k}\rangle\end{array}\right\}\Rightarrow|\psi\rangle = |\mathbf{k}\rangle + \widehat{G}^+\widehat{v}\,|\psi\rangle$$

$$\widehat{G}^+ := \left(-\widehat{\mathbf{p}}^2 + k^2 + i\epsilon\right)^{-1}$$

Lippmann-Schwinger eq./Green's fn. method:

$$\langle \mathbf{x}|\psi\rangle = \langle \mathbf{x}|\mathbf{k}\rangle + \langle \mathbf{x}|\widehat{G}^{+}\widehat{v}\,|\psi\rangle$$

$$\langle \mathbf{x}|\widehat{G}^{+}|\mathbf{x}\,'\rangle := -\frac{i}{4}H_0^{(1)}(k|\mathbf{x}-\mathbf{x}\,'|)$$

Lippmann-Schwinger eq./Green's fn. method:

$$\langle \mathbf{x}|\psi\rangle = \langle \mathbf{x}|\mathbf{k}\rangle + \langle \mathbf{x}|\widehat{G}^{+}\widehat{v}\,|\psi\rangle \quad \to \frac{1}{2\pi}\left[e^{i\mathbf{k}\cdot\mathbf{x}} + \frac{e^{ikr}}{\sqrt{r}} f(\mathbf{k}',\mathbf{k})\right]$$

$$\langle \mathbf{x}|\widehat{G}^{+}|\mathbf{x}\,'\rangle := -\frac{i}{4} H_0^{(1)}(k|\mathbf{x}-\mathbf{x}\,'|) \qquad \text{as} \quad r := |\mathbf{x}| \to \infty$$

$$\mathbf{k}' := \frac{k\mathbf{x}}{r}$$

$f(\mathbf{k}',\mathbf{k}) := -\pi\sqrt{2\pi i/k}\,\langle \mathbf{k}'|\widehat{v}|\psi\rangle$: Scattering amplitude

Lippmann-Schwinger eq./Green's fn. method:

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$$|\psi\rangle = |\mathbf{k}\rangle + \widehat{G}^{+}\widehat{v}\,|\psi\rangle$$

$$\Rightarrow |\psi\rangle = [1 - \widehat{G}^{+}\widehat{v}]^{-1}|\mathbf{k}\rangle = \sum_{n=0}^{\infty}(\widehat{G}^{+}\widehat{v})^{n}|\mathbf{k}\rangle$$

Lippmann-Schwinger eq./Green's fn. method:

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$$f_{\ell}(\mathbf{k}',\mathbf{k}) := -\pi\sqrt{2\pi i/k}\langle \mathbf{k}'|\widehat{v}\,(\widehat{G}^{+}\,\widehat{v})^{\ell-1}|\mathbf{k}\rangle$$

Lippmann-Schwinger eq./Green's fn. method:

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N-th Born approximation: $f(\mathbf{k}',\mathbf{k}) \approx \sum_{\ell=1}^{N} f_{\ell}(\mathbf{k}',\mathbf{k})$

Multi-delta-function potential:

$$v(\mathbf{x}) = \sum_{j=1}^{N} \mathfrak{z}_j \, \delta(\mathbf{x} - \mathbf{a}_j), \qquad \mathbf{x} \in \mathbb{R}^d, \;\; d \in \{2, 3\}$$

$$\Rightarrow \quad \widehat{v} := \sum_{j=1}^{N} \mathfrak{z}_j \, |\mathbf{a}_j\rangle\langle\mathbf{a}_j|$$

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Lippmann-Schwinger eq.: $|\psi\rangle = |\mathbf{k}\rangle + \widehat{G}\,\hat{v}|\psi\rangle$

$$\Rightarrow \quad \psi(\mathbf{x}) = \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{d/2}} + \sum_{n=1}^{N} G(\mathbf{x} - \mathbf{a}_n) X_n \qquad (\star)$$

$$G(\mathbf{x} - \mathbf{x}') := \langle\mathbf{x}|\widehat{G}|\mathbf{x}'\rangle, \qquad X_n := \mathfrak{z}_n \psi(\mathbf{a}_n)$$

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$$(\star) \quad \Rightarrow \quad \sum_{n=1}^{N} A_{mn} X_n = \frac{e^{i\mathbf{a}_m\cdot\mathbf{k}}}{(2\pi)^{d/2}}$$

$$A_{mn} := \mathfrak{z}_n^{-1} \delta_{mn} - G(\mathbf{a}_m - \mathbf{a}_n)$$

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$$\psi(\mathbf{x}) \to \frac{1}{(2\pi)^{d/2}} \left[e^{i\mathbf{k}\cdot\mathbf{x}} + \frac{\mathfrak{f}(\mathbf{k}', \mathbf{k})\, e^{ikr}}{r^{\frac{d-1}{2}}} \right] \quad \text{for} \quad r \to \infty$$

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$$G(\mathbf{x}) = \begin{cases} -\frac{i}{4} H_0^{(1)}(kr) & \text{for} \quad d = 2 \\ -\frac{1}{4\pi} \frac{e^{ikr}}{r} & \text{for} \quad d = 3 \end{cases}$$

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$$G(\mathbf{x} - \mathbf{x}') \to \left\{ \begin{array}{ll} -\sqrt{\dfrac{i}{8\pi kr}}\, e^{-i\mathbf{k}'\cdot\mathbf{x}'} e^{ikr} & \text{for} \quad d = 2 \\ -\dfrac{1}{4\pi} \dfrac{e^{-i\mathbf{k}'\cdot\mathbf{x}'} e^{ikr}}{r} & \text{for} \quad d = 3 \end{array} \right\} \quad \text{for} \quad r := |\mathbf{x}| \to \infty$$

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$$\mathfrak{f}(\mathbf{k}', \mathbf{k}) = \frac{c_d}{\sqrt{k^{3-d}}} \sum_{m,n=1}^{N} A^{-1}_{mn}\, e^{i(\mathbf{a}_n\cdot\mathbf{k} - \mathbf{a}_m\cdot\mathbf{k}')}$$

$$c_d := \begin{cases} -\sqrt{i/8\pi} & \text{for} \quad d = 2 \\ -1/4\pi & \text{for} \quad d = 3 \end{cases}$$

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But : $G(\mathbf{0}) = \infty \quad \Rightarrow \quad A_{mm} = \mathfrak{z}_m^{-1} - G(\mathbf{0}) = \infty$!

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But : $G(\mathbf{0}) = \infty \quad \Rightarrow \quad A_{mm} = \mathfrak{z}_m^{-1} - G(\mathbf{0}) = \infty$!

Absorb the singularity of $G(\mathbf{x})$ at $\mathbf{x} = \mathbf{0}$ in $\mathfrak{z}_m^{-1}$.

Renormalize: $\mathfrak{z}_m \to \tilde{\mathfrak{z}}_m$

$$\mathfrak{f}(\mathbf{k}', \mathbf{k}) = \frac{c_d}{\sqrt{k^{3-d}}} \sum_{m,n=1}^{N} A_{mn}^{-1}\, e^{i(\mathbf{a}_n \cdot \mathbf{k} - \mathbf{a}_m \cdot \mathbf{k}')}$$

$$d = 2\text{:}\quad A_{mn} = \begin{cases} \tilde{\mathfrak{z}}_m^{-1} + \frac{i}{4} & \text{for } m = n \\ \frac{i}{4}\, H_0^{(1)}(k|\mathbf{a}_m - \mathbf{a}_n|) & \text{for } m \neq n \end{cases}$$

$$d = 3\text{:}\quad A_{mn} = \begin{cases} \tilde{\mathfrak{z}}_n^{-1} + \frac{ik}{4\pi} & \text{for } m = n \\ \dfrac{e^{ik|\mathbf{a}_m - \mathbf{a}_n|}}{4\pi|\mathbf{a}_m - \mathbf{a}_n|} & \text{for } m \neq n \end{cases}$$

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$$d=3\!:\; A_{mn} = \begin{cases} \tilde{\mathfrak{z}}_n^{-1} + \frac{ik}{4\pi} & \text{for } m=n \\ \frac{e^{ik|\mathbf{a}_m-\mathbf{a}_n|}}{4\pi|\mathbf{a}_m-\mathbf{a}_n|} & \text{for } m\neq n \end{cases}$$

$N=1$: $v(\mathbf{x}) = \mathfrak{z}_1\delta(\mathbf{x}-\mathbf{a}_1)$

$$\mathfrak{f}(\mathbf{k}',\mathbf{k}) = \begin{cases} -\sqrt{\frac{i}{8\pi k}}\,\frac{e^{i\mathbf{a}_1\cdot(\mathbf{k}-\mathbf{k}')}}{\tilde{\mathfrak{z}}_1^{-1}+\frac{i}{4}} & \text{for}\quad d=2 \\ -\frac{1}{4\pi}\,\frac{e^{i\mathbf{a}_1\cdot(\mathbf{k}-\mathbf{k}')}}{\tilde{\mathfrak{z}}_1^{-1}+\frac{ik}{4\pi}} & \text{for}\quad d=3 \end{cases}$$

[Jackiw 1991]

$$\mathfrak{f}(\mathbf{k}',\mathbf{k}) = \frac{c_d}{\sqrt{k^{3-d}}} \sum_{m,n=1}^{N} A_{mn}^{-1}\, e^{i(\mathbf{a}_n\cdot\mathbf{k}-\mathbf{a}_m\cdot\mathbf{k}')}$$

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Coincidence-limit problem:

$\mathbf{a}_i \to \mathbf{a}_j$ for some i and j $\Rightarrow \mathfrak{f}(\mathbf{k}',\mathbf{k}) \to 0\,!$

$$\mathfrak{f}(\mathbf{k}',\mathbf{k}) = \frac{c_d}{\sqrt{k^{3-d}}} \sum_{m,n=1}^{N} A^{-1}_{mn}\, e^{i(\mathbf{a}_n\cdot\mathbf{k}-\mathbf{a}_m\cdot\mathbf{k}')}$$

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Coincidence-limit problem:

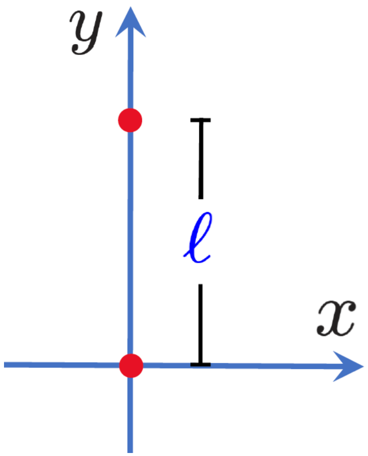
$$\mathbf{a}_i \to \mathbf{a}_j \text{ for some } i \text{ and } j \Rightarrow \mathfrak{f}(\mathbf{k}',\mathbf{k}) \to 0\,!$$

WOLOG: Set $i = N-1$ and $j = N$. Then $\mathbf{a}_i \to \mathbf{a}_j$ implies

$$v(\mathbf{x}) = \sum_{n=1}^{N} \mathfrak{z}_n \delta(\mathbf{x}-\mathbf{a}_n) \to \sum_{n=1}^{N-2} \mathfrak{z}_n \delta(\mathbf{x}-\mathbf{a}_n) + (\mathfrak{z}_{N-1} + \mathfrak{z}_N)\delta(\mathbf{x}-\mathbf{a}_{n-1})$$

In **2D**, WOLOG

N=2: $v(x,y) = \mathfrak{z}_1\,\delta(x)\delta(y) + \mathfrak{z}_2\,\delta(x)\delta(y-\ell),\ \ell \in \mathbb{R}^+$



In **2D**, WOLOG

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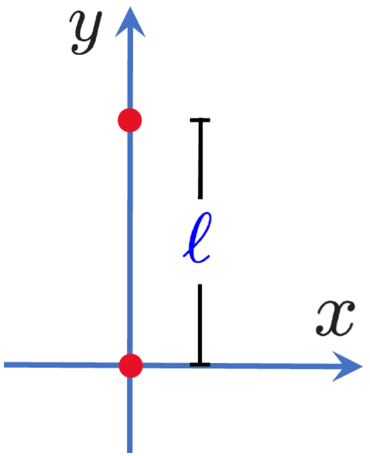
$$\mathfrak{f}(\mathbf{k}',\mathbf{k}) = -\sqrt{\frac{i}{8\pi k}}\left[\mathfrak{f}_1(\mathbf{k}) + \mathfrak{f}_2(\mathbf{k})e^{-ik\ell\,\mathsf{sin}\,\theta}\right]$$



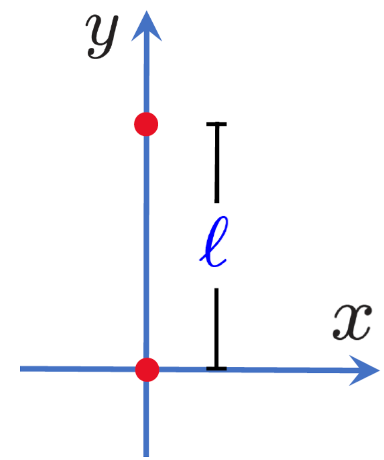
$$\mathfrak{f}_1(\mathbf{k}) := \frac{4\left[4\tilde{\mathfrak{z}}_2{}^{-1} + i - iH_0^{(1)}(k\ell)e^{ik\ell\,\mathsf{sin}\,\theta_0}\right]}{(4\tilde{\mathfrak{z}}_1{}^{-1}+i)(4\tilde{\mathfrak{z}}_2{}^{-1}+i) + H_0^{(1)}(k\ell)^2}$$

$$\mathfrak{f}_2(\mathbf{k}) := \frac{4\left[(4\tilde{\mathfrak{z}}_1{}^{-1} + i)e^{ik\ell\,\mathsf{sin}\,\theta_0} - iH_0^{(1)}(k\ell)\right]}{(4\tilde{\mathfrak{z}}_1{}^{-1}+i)(4\tilde{\mathfrak{z}}_2{}^{-1}+i) + H_0^{(1)}(k\ell)^2}$$

In **2D**, WOLOG

N=2: $v(x,y) = \mathfrak{z}_1\,\delta(x)\delta(y) + \mathfrak{z}_2\,\delta(x)\delta(y-\ell),\ \ell \in \mathbb{R}^+$

$$\mathfrak{f}(\mathbf{k}',\mathbf{k}) = -\sqrt{\frac{i}{8\pi k}}\left[\mathfrak{f}_1(\mathbf{k}) + \mathfrak{f}_2(\mathbf{k})e^{-ik\ell\sin\theta}\right]$$



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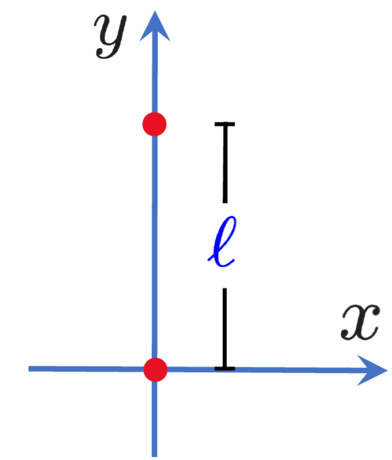

For $\ell \to 0$, $\quad v(x,y) \to \mathfrak{z}\,\delta(x)\delta(y), \quad \mathfrak{z} := \mathfrak{z}_1 + \mathfrak{z}_2,$

$$\mathfrak{f}(\mathbf{k}',\mathbf{k}) \to -\sqrt{\frac{i}{8\pi k}}\,\frac{1}{\tilde{\mathfrak{z}}^{-1} + \frac{i}{4}}$$

Dynamical formulation of stationary scattering (DFSS)

1D: Schrödinger eq.: $[-\partial_x^2 + v(x)]\psi(x) = k^2\psi(x)$

$v : \mathbb{R} \to \mathbb{C}$ is a short-range potential,

$$x\, v(x) \to 0 \quad \text{for} \quad x \to \pm\infty$$

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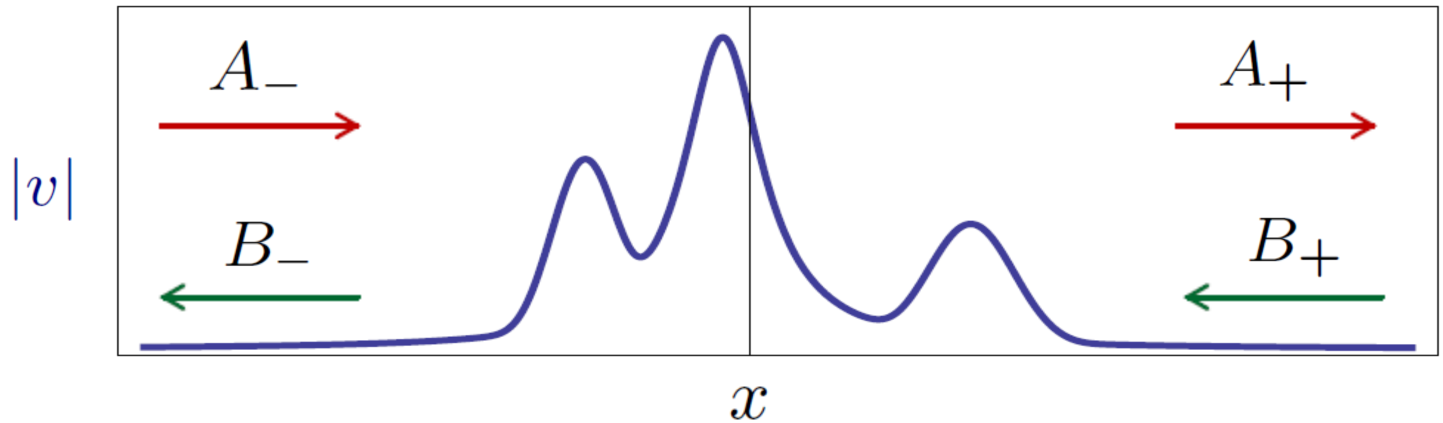

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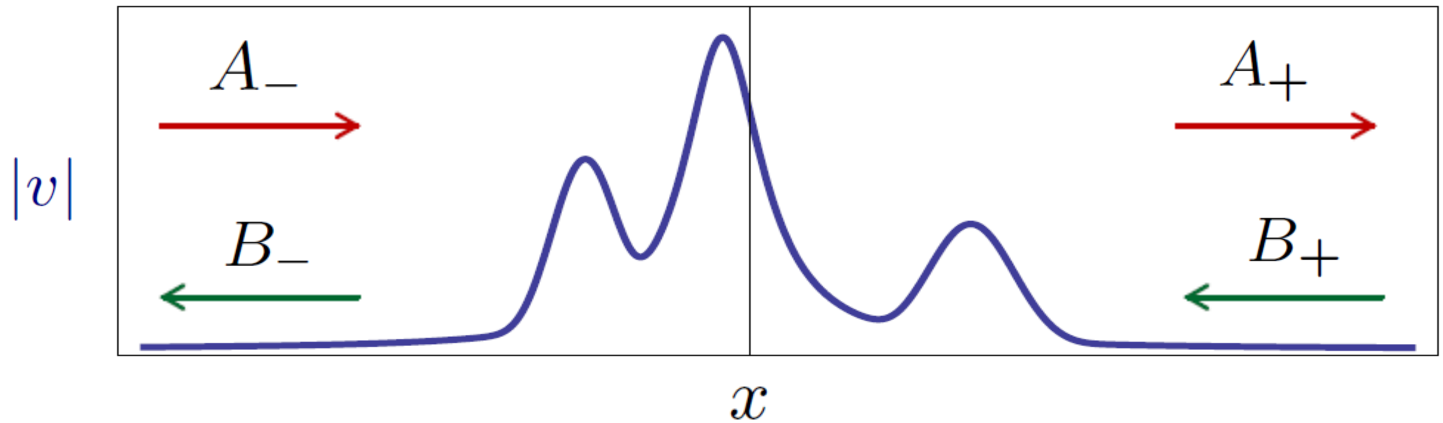

- Transfer matrix: $\begin{bmatrix} A_+ \\ B_+ \end{bmatrix} = \mathbf{M} \begin{bmatrix} A_- \\ B_- \end{bmatrix}.$

- Scattering from the left and right:

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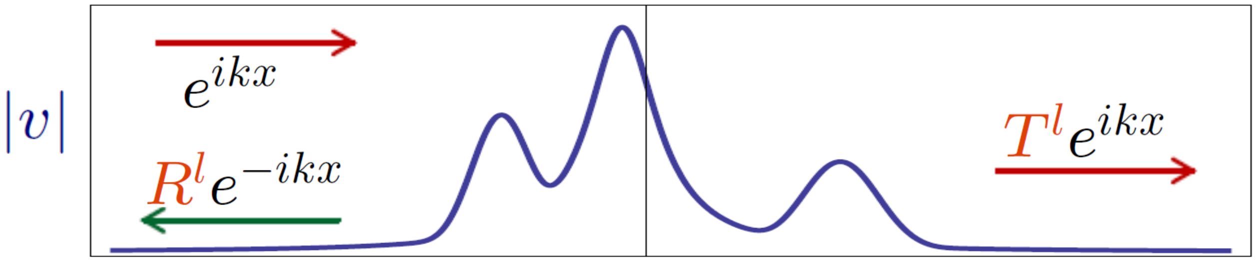

- Scattering from the left and right:

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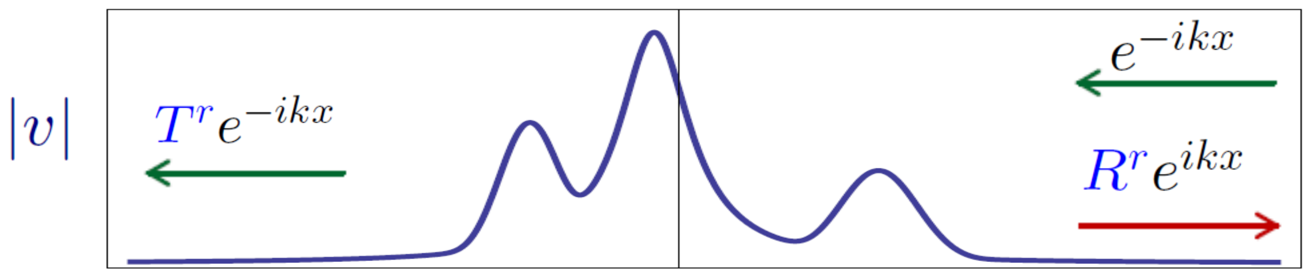

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$$R^l = -\frac{M_{21}}{M_{22}}, \qquad R^r = \frac{M_{12}}{M_{22}}, \qquad T^l = T^r =: T = \frac{1}{M_{22}}.$$

Theorem: $\mathbf{M} = \mathbf{U}(+\infty, -\infty)$ where $\mathbf{U}(x, x_0)$ is the evolution operator for

$$\mathbf{H}(x) := \frac{v(x)}{2k} \begin{bmatrix} 1 & e^{-2ikx} \\ -e^{2ikx} & -1 \end{bmatrix}.$$

---

x plays the role of "time".

$$i\partial_x \mathbf{U}(x, x_0) = \mathbf{H}(x)\mathbf{U}(x, x_0)$$

[Ann. Phys. (NY), **341**, 77 (2014)]

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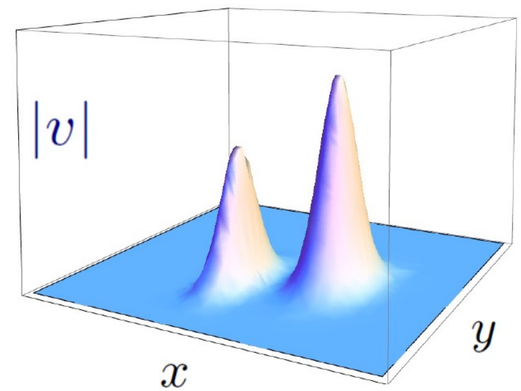
i.e.,

$$\begin{aligned} \mathbf{M} =& \mathscr{T} \exp \int_{-\infty}^{\infty} -i\mathbf{H}(x)dx \\ =& \mathbf{I} - i \int_{-\infty}^{\infty} dx_1 \mathbf{H}(x_1) \\ & + (-i)^2 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{x_2} dx_1 \mathbf{H}(x_2)\mathbf{H}(x_1) + \cdots \end{aligned}$$

[Ann. Phys. (NY), **341**, 77 (2014)]

2D:

$$\left[-\partial_x^2 - \partial_y^2 + v(x,y)\right]\psi(x,y) = k^2\psi(x,y)$$



$\mathscr{F}$: Vector space of functions $\phi : \mathbb{R} \to \mathbb{C}$

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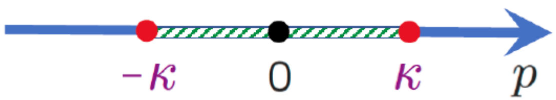

[PRA **93**, 042707 (2016)]

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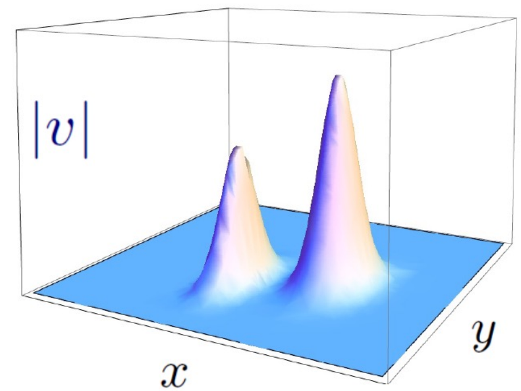

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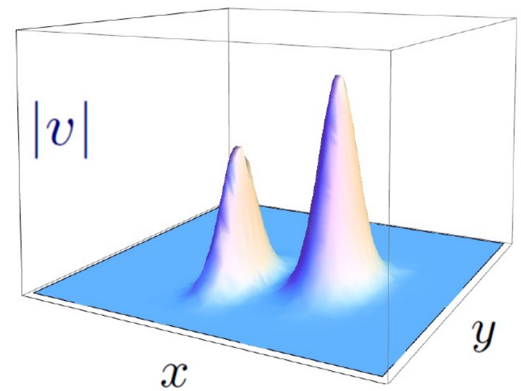

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$$\widehat{\mathbf{M}} : \mathscr{F}_k^2 \to \mathscr{F}_k^2$$

[PRA **93**, 042707 (2016)]

- $\widehat{\mathrm{M}}$ **admits a formal Dyson series expansion:**

$$\begin{aligned}\widehat{\mathrm{M}} &= \widehat{\mathbf{\Pi}}_k \mathscr{T} \exp\left\{-i\int_{-\infty}^{\infty} dx \widehat{\boldsymbol{\mathcal{H}}}(x)\right\}\widehat{\mathbf{\Pi}}_k \\ &= \widehat{\mathbf{\Pi}}_k - i\int_{-\infty}^{\infty} dx_1 \widehat{\mathbf{\Pi}}_k \widehat{\boldsymbol{\mathcal{H}}}(x_1)\widehat{\mathbf{\Pi}}_k \\ &\quad - \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{x_2} dx_1 \widehat{\mathbf{\Pi}}_k \widehat{\boldsymbol{\mathcal{H}}}(x_2)\widehat{\boldsymbol{\mathcal{H}}}(x_1)\widehat{\mathbf{\Pi}}_k + \cdots\end{aligned}$$

Projection onto $\mathscr{F}_k^2$: $(\widehat{\mathbf{\Pi}}_k F)(p) := \begin{cases} F(p) & \text{for} \quad p \in (-k, k) \\ 0 & \text{for} \quad p \notin (-k, k) \end{cases}$

[PRA **104**, 03222 (2021)]

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$$\widehat{\boldsymbol{\mathcal{H}}}(x) := \frac{1}{2} e^{-i\widehat{\varpi}x\boldsymbol{\sigma}_3}\, v(x,\widehat{y})\, \widehat{\varpi}^{-1} \boldsymbol{\mathcal{K}}\, e^{i\widehat{\varpi}x\boldsymbol{\sigma}_3},$$

$$(\widehat{\varpi}f)(p) := \varpi(p)\, f(p), \qquad \boldsymbol{\sigma}_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$(\widehat{y}f)(p) := i\partial_p f(p), \qquad \boldsymbol{\mathcal{K}} := \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

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[PRA **104**, 03222 (2021)]

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$$= \frac{1}{2}\begin{bmatrix} e^{-ix\,\widehat{\varpi}}\,\widehat{u}(x)\,e^{ix\,\widehat{\varpi}} & e^{-ix\,\widehat{\varpi}}\,\widehat{u}(x)\,e^{-ix\,\widehat{\varpi}} \\ -e^{ix\,\widehat{\varpi}}\,\widehat{u}(x)\,e^{ix\,\widehat{\varpi}} & -e^{ix\,\widehat{\varpi}}\,\widehat{u}(x)\,e^{-ix\,\widehat{\varpi}} \end{bmatrix}$$

$$\widehat{u}(x) := v(x,\widehat{y})\widehat{\varpi}^{-1}$$

[PRA **104**, 03222 (2021)]

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$$\boxed{\big(v(x,\widehat{y})f\big)(p) := \frac{1}{2\pi}\int_{-\infty}^{\infty} dq\, \tilde{v}(x,p-q)f(q)}$$

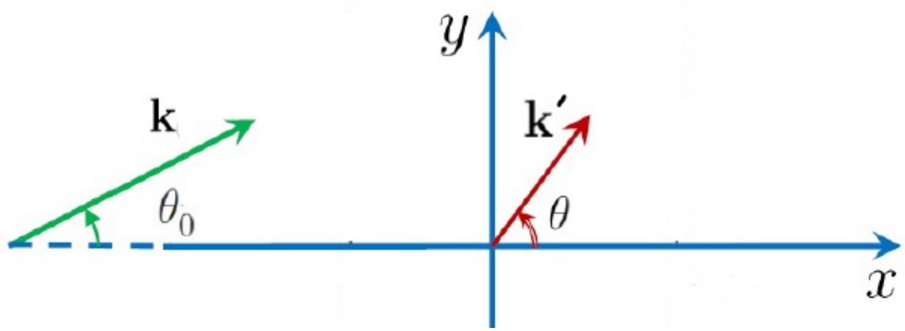
$$\tilde{v}(x,p) := \int_{-\infty}^{\infty} dy\, e^{-ipy} v(x,y)$$

[PRA **104**, 03222 (2021)]

- $\widehat{\mathrm{M}}$ **can be used to compute** $f(\mathbf{k}',\mathbf{k})$.

$$f(\mathbf{k}',\mathbf{k}) = \frac{e^{-i\pi/4}}{\sqrt{2\pi k}} \times \begin{cases} A_+(k\sin\theta) - 2\pi\delta(\theta-\theta_0) & \text{for} \quad \cos\theta > 0 \\ B_-(k\sin\theta) & \text{for} \quad \cos\theta < 0 \end{cases}$$

Left incidence: $\theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$



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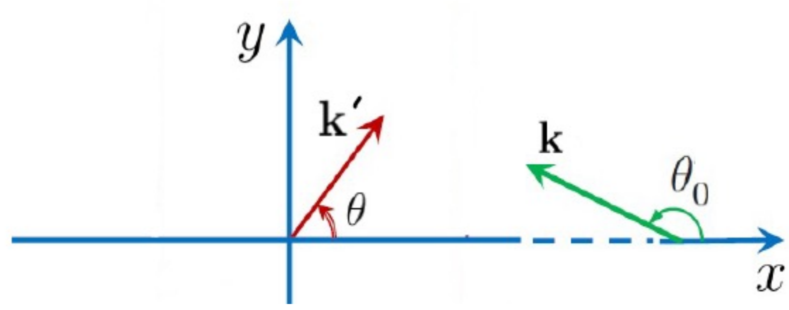

[PRA **104**, 03222 (2021)]

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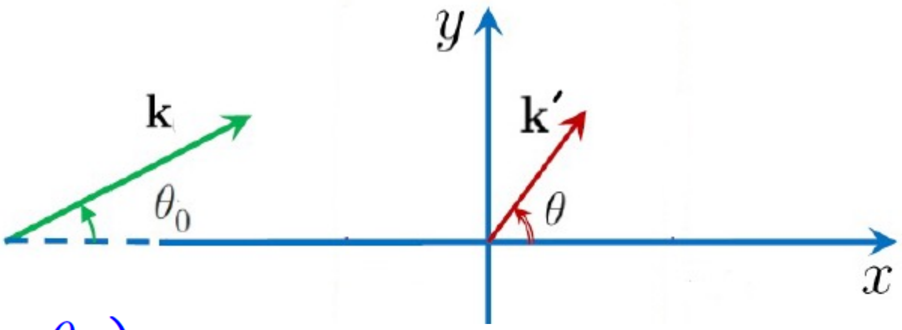

$$\Delta(p) := 2\pi k \cos\theta_0\, \delta(p - k\sin\theta_0)$$

[PRA **104**, 03222 (2021)]

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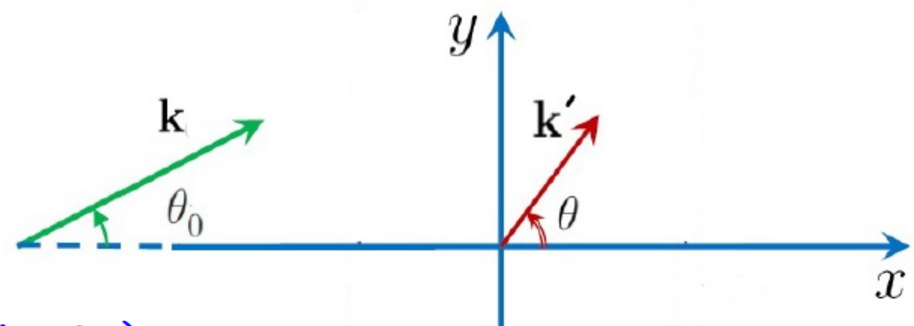

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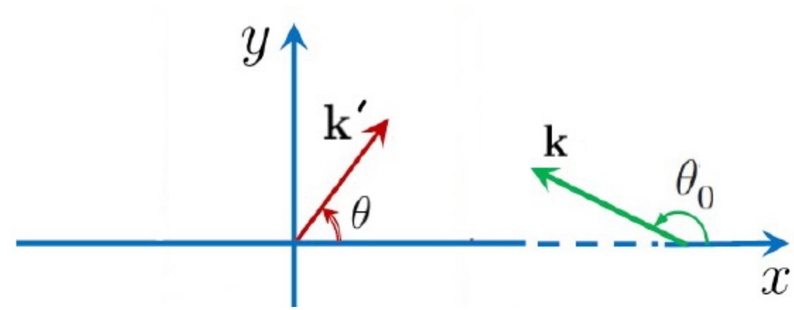

[PRA **104**, 03222 (2021)]

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$$= \frac{1}{2}\,\delta(x) g(\widehat{y})\, \widehat{\varpi}^{-1}\boldsymbol{\mathcal{K}}$$

[Ann. Phys. **443**, 168966 (2022), arXiv: 2204.09554]

For $v(x,y) = g(y)\delta(x)$, $\quad \widehat{\mathcal{H}}(x) = \frac{1}{2}\delta(x) g(\widehat{y})\,\widehat{\varpi}^{-1}\mathcal{K}$

$$\mathcal{K} := \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow \mathcal{K}^2 = \mathbf{0} \Rightarrow \widehat{\mathcal{H}}(x_1)\widehat{\mathcal{H}}(x_2) = \widehat{\mathbf{0}}$$

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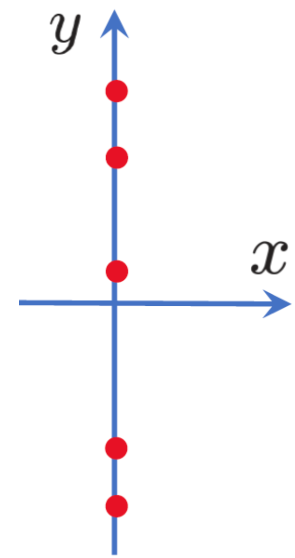
For

$$v(x,y) = \sum_{n=1}^{N} \mathfrak{z}_n \delta(x)\delta(y-a_n)$$

we can use $\widehat{M_{ij}}$ to compute $\mathfrak{f}(\mathbf{k}',\mathbf{k})$.



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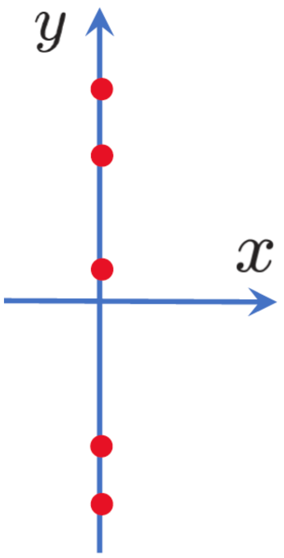
This again gives

$$\mathfrak{f}(\mathbf{k}', \mathbf{k}) = -\sqrt{\frac{i}{8\pi k}} \sum_{m,n=1}^{N} A_{mn}^{-1}\, e^{i(\mathbf{a}_n\cdot\mathbf{k} - \mathbf{a}_m\cdot\mathbf{k}')},$$



but now

$$A_{mn} := \begin{cases} \mathfrak{z}_m^{-1} + \frac{i}{4} & \text{for} \quad m = n \\ \frac{i}{4} J_0(k|a_m - a_n|) & \text{for} \quad m \neq n \end{cases}$$

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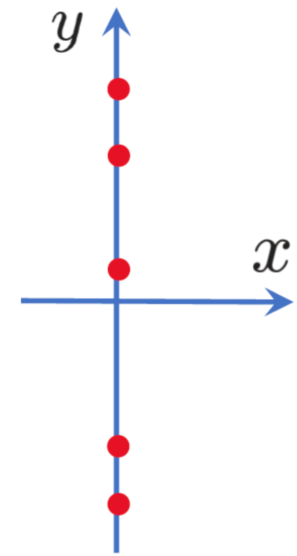

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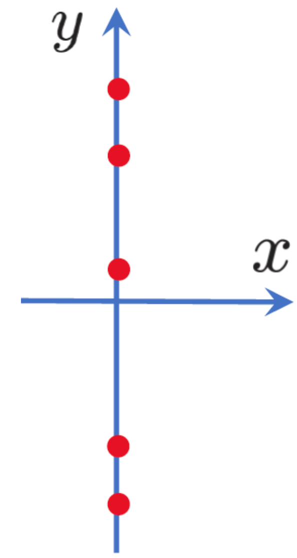

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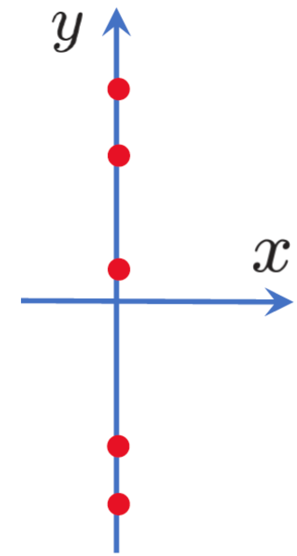

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- **This formula has correct coincidence limit.**

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Again nothing blows up, so **no need for renormalizing $\mathfrak{z}_n$**, and we obtain the **correct coincidence limit**.

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Exactness of the N-th-order Born Approximation

$$\widehat{\mathcal{H}}(x) := \frac{1}{2}\, e^{-i\widehat{\varpi}x\boldsymbol{\sigma}_3}\, v(x,\widehat{y})\, \widehat{\varpi}^{-1}\mathcal{K}\, e^{i\widehat{\varpi}x\boldsymbol{\sigma}_3}$$

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Qxn: Are there potentials for which this series truncates?

Born series: $$f(\mathbf{k}',\mathbf{k}) = \sum_{\ell=1}^{\infty} f_\ell(\mathbf{k}',\mathbf{k})$$

$$f_\ell(\mathbf{k}',\mathbf{k}) := -\pi\sqrt{2\pi}\langle\,\mathbf{k}'|\hat{v}\,(\hat{G}^{+}\,\hat{v})^{\ell-1}|\mathbf{k}\rangle$$

Theorem: Let $d \in \{2,3\}$ and $v : \mathbb{R}^d \to \mathbb{C}$ be a short-range potential with Fourier transform $\widetilde{v} : \mathbb{R}^d \to \mathbb{C}$. Suppose that

$$\widetilde{v}(\mathbf{p}) = 0 \quad \text{for} \quad \hat{\mathbf{u}} \cdot \mathbf{p} < \alpha$$

for some $\alpha \in \mathbb{R}^+$ and $\hat{\mathbf{u}} \in \mathbb{R}^d$ such that $|\hat{\mathbf{u}}| = 1$.

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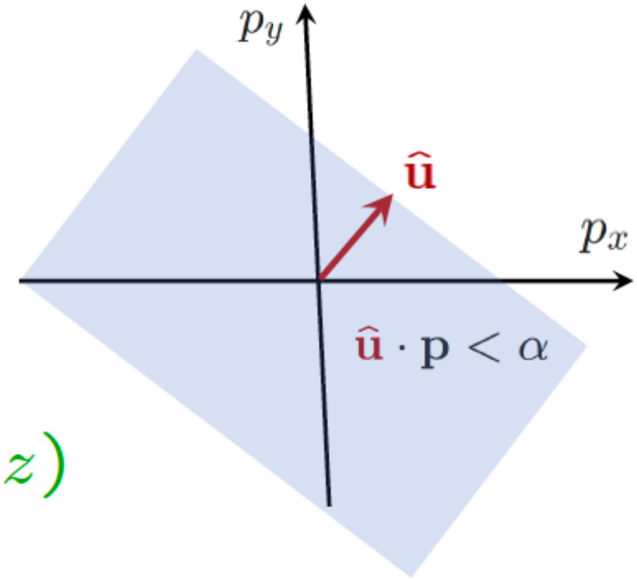

For $\hat{\mathbf{u}}$ along x axis,

$$v(x,y,z) = e^{i\alpha x} \int_0^\infty d\mathfrak{K}\, e^{i\mathfrak{K}x} f(\mathfrak{K}, y, z)$$

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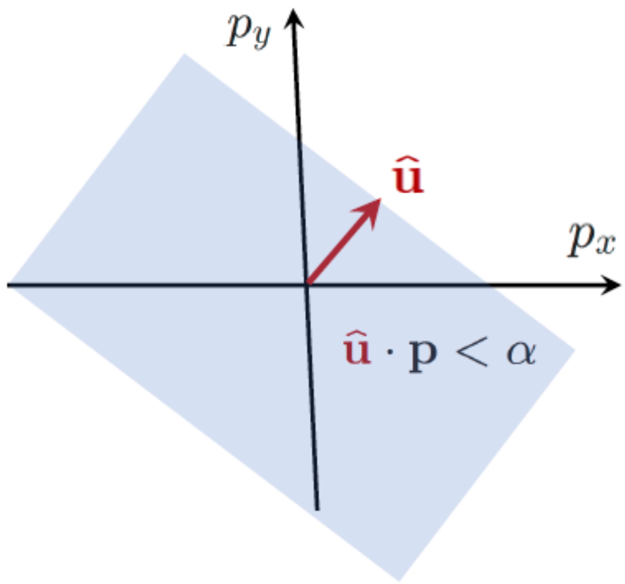
$$N := \lfloor 2k/\alpha \rfloor$$

i.e., N is the largest integer that is not larger than $2k/\alpha$, so that $k < \frac{1}{2}(N+1)\alpha$.



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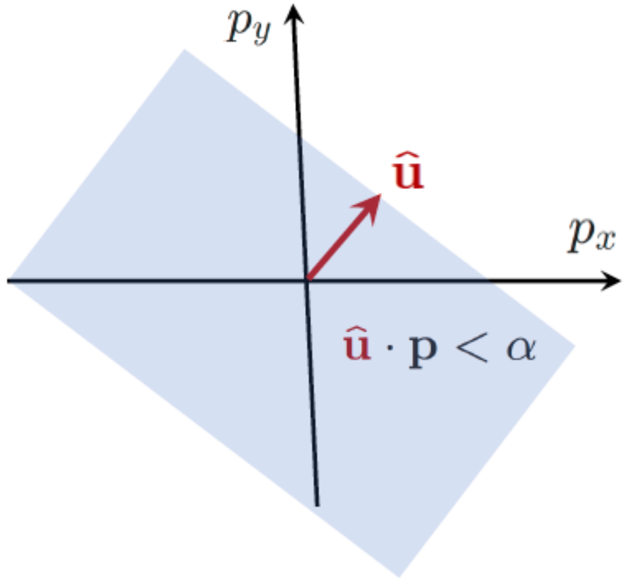

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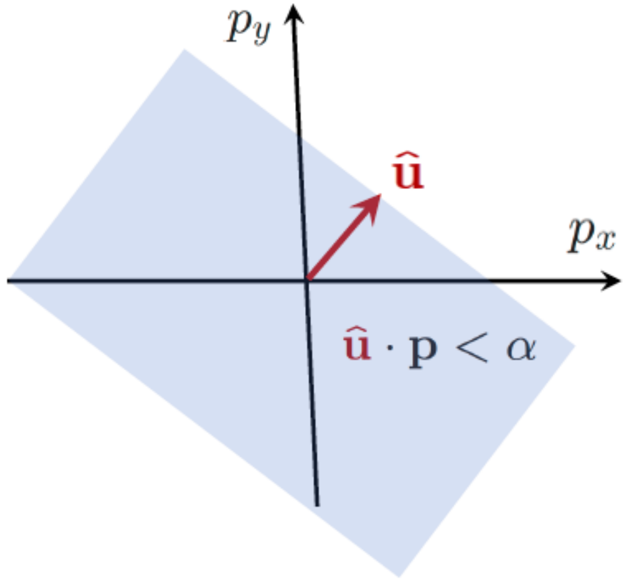

In particular, v **does not scatter waves with** $k < \alpha/2$, and the **1st Born approximation is exact for** $k < \alpha$.

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Concluding Remarks:

- Standard treatment of multi-delta-function point scatterers, which involves renormalization of coupling-constants, suffers from the coincidence-limit problem.

- This can be remedied if one allows the renormalized coupling constants also depend on the distances between the point scatterers, but the method does not give a clue as to how this dependence should be.

- DFSS on the contrary produces finite results with a consistent coincidence limit.

- DFSS has found a number of other remarkable applications such as the discovery of complex potentials for which the N-th Born approximation is exact. This result also extends to EM scattering.

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Thank you for your attention.