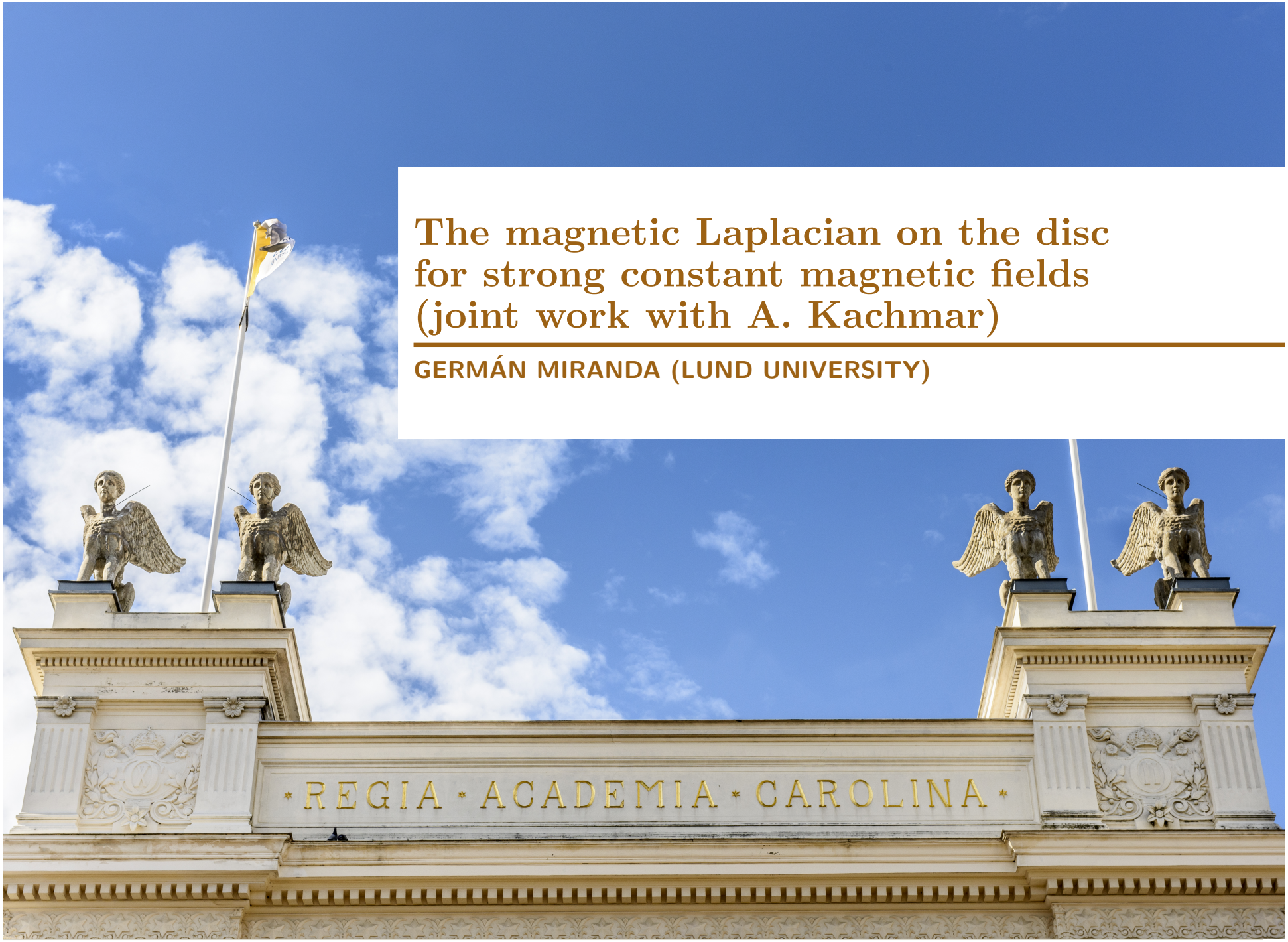
The magnetic Laplacian on the disc for strong constant magnetic fields (joint work with A. Kachmar)

GERMÁN MIRANDA (LUND UNIVERSITY)



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The magnetic Laplacian on the disc

Let $b > 0$ strength of the magnetic field and

$$H_b(R) = (-i\nabla + bA)^2,$$

where $A(x_1, x_2) = \frac{1}{2}(-x_2, x_1)$ generating constant magnetic field $\vec{B} = (0, 0, 1)$. Domain $\Omega_R := D(0, R) = \{x \in \mathbb{R}^2 : |x| < R\}$.

- Dirichlet $u|_{\partial\Omega_R} = 0$
- Neumann $\partial_r u|_{\partial\Omega_R} = 0$

$H_b(R)$ is unitarily equivalent to $R^{-2}H_{bR^2}(1) \Rightarrow$ we study $H_b(1)$

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Decomposing $u \in L^2(\Omega)$ as $u(r, \theta) = \sum_{m\in\mathbb{Z}} u_m(r)\frac{e^{-im\theta}}{\sqrt{2\pi}}$ we can rewrite

$$H_b = \bigoplus_{m\in\mathbb{Z}} \Big(H_{m,b} \otimes I_m \Big),$$

with

$$H_{m,b} = -\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \left(\frac{m}{r} - \frac{br}{2}\right)^2,$$

acting on $L^2((0, 1), r\,dr)$ with

- Dirichlet $u(1) = 0$
- Neumann $u'(1) = 0$

Study of the critical field H_{C_3} in a cylindrical geometry by D. Saint-James (1965)

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Infinite cylinder Ω_R of radius R, the linearized Ginzburg–Landau equation is given by

$$\frac{1}{2m}\left[-i\hbar\nabla-\frac{2eA}{c}\right]^2 u+\alpha u=0,$$

with boundary condition

$$\left(-i\hbar\nabla-\tfrac{2eA}{c}\right)u|_{\partial\Omega_R}=0.$$

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Cylindrical coord $\Rightarrow$ Study of $H^N_{m,b}$.

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- Special functions. Hard!
- Numerical computations (*Calculateur électronique*)

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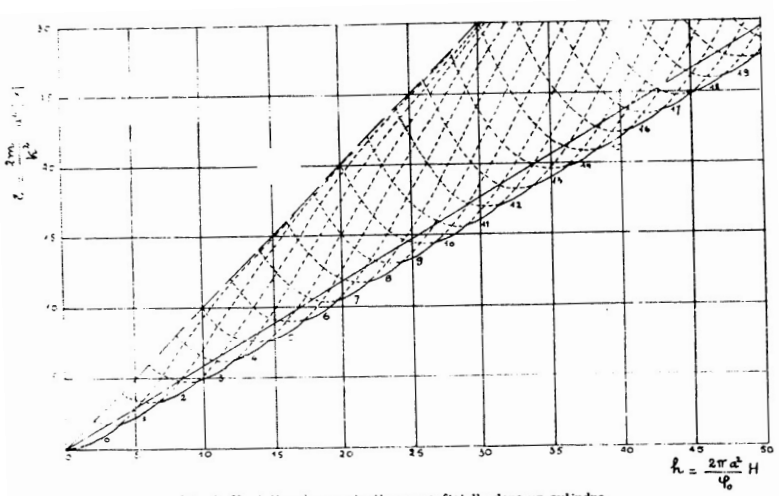

Fig. 1. Variation de germination superficielle dans un cylindre.

$$H_{C_2}=\frac{m|\alpha|}{e\hbar}$$

$$H_{C_3}\sim\frac{\hbar}{2eR^2}f^{-1}\left(\frac{2mR^2|\alpha|}{\hbar^2}\right)$$

Example:

$$f(h)=\Theta_0 h\Rightarrow H_{C_3}\sim\frac{H_{C_2}}{\Theta_0}.$$

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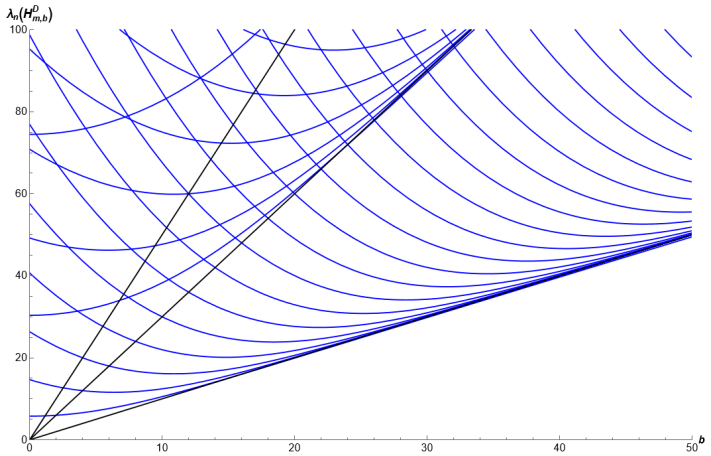
Dirichlet case can be reduced to finding roots of special functions or using variational methods.



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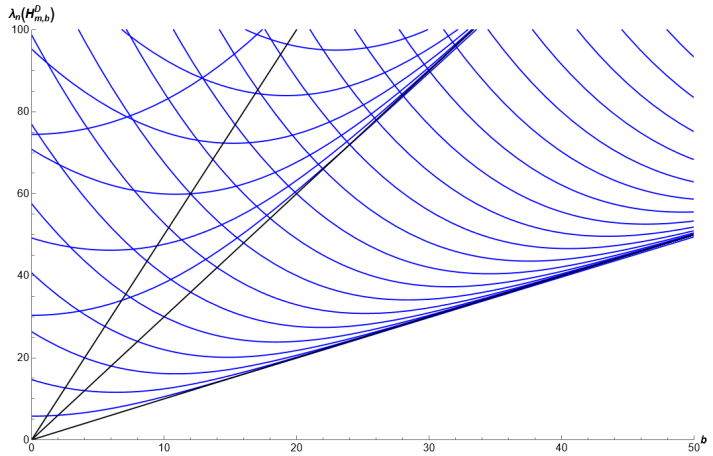

- 1996 Erdös $n = 1, m = 0$ and $\epsilon > 0$

$$b + \frac{C_1(\epsilon)}{bR^2} e^{-(\frac{1}{2}+\epsilon)bR^2} \leq \lambda_1(H^D_{1,b}(R)) \leq b + C_2\left(\frac{1}{bR^2} + bR^2\right) e^{-\frac{bR^2}{8}}.$$

Upper bound: trial state $e^{-\frac{r^2}{4}}$ and suitable cut off.
Lower bound: Birman–Schwinger principle.

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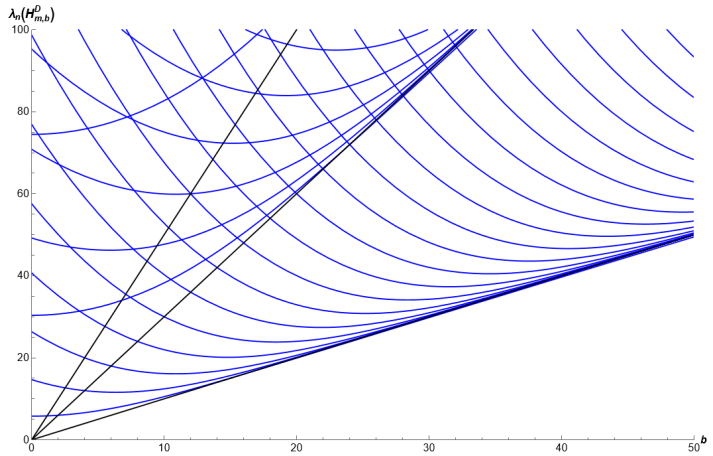

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- 2017 Helffer and Sundqvist $n = 1$ and $m \geq 0$

$$\lambda_1(H^D_{m,b}(R)) = b + \frac{b^{m+2} R^{2(1+m)}}{2^m m!} e^{-\frac{bR^2}{2}} (1 + \mathcal{O}(b^{-1})).$$

  Upper bound: trial state $u_m(r) = r^m (e^{\frac{b}{4}(R^2 - r^2)} - e^{-\frac{bR^2}{4}(R^2 - r^2)})$.
  Lower bound: Temple inequality.

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- 2024 Baur and Weidl, $n \in \mathbb{N}$ and $m \in \mathbb{Z}$

$$\lambda_n(H^D_{m,b}(R)) = (2n-1+|m|-m)b + e^{-\frac{bR^2}{2}} \frac{b^{2n+m} R^{2(2n+m-1)}}{(n-1)!(m+n-1)!2^{2(n-1)+m}}(1+\mathcal{O}(b^{-1})).$$

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Idea of the proof: Reducing the problem to find λ solving the implicit equation

$$M\left(\frac{1}{2}\left(|m|-m+1-\frac{\lambda}{b}\right), |m|+1, \frac{bR^2}{2}\right) = 0,$$

where $M(a,b,z)$ is Kummer's confluent hypergeometric function.

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For Neumann is more complicated, for $m \geq 0$ we have

$$2bM'\left(\frac{1}{2}\left(1-\frac{\lambda}{b}\right), m+1, \frac{b}{2}\right) - (b-2m)M\left(\frac{1}{2}\left(1-\frac{\lambda}{b}\right), m+1, \frac{b}{2}\right) = 0.$$

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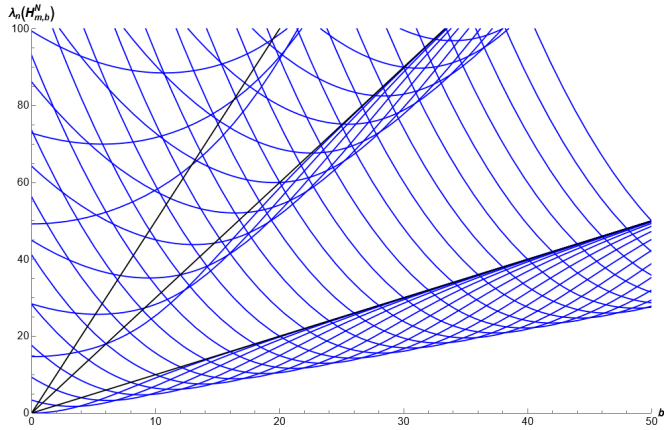

Let

$$H^N_{m,b} = -\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \left(\frac{m}{r} - \frac{br}{2}\right)^2$$

with

$$\mathcal{D}(H^N_{m,b}) = \{u\colon u, u/r, u', H^N_{m,b}u \in L^2((0,1), r\,dr),\ u'(1) = 0\},$$

for $m \neq 0$, and

$$\mathcal{D}(H^N_{0,b}) = \{u\colon u, H_{0,b}u \in L^2((0,1), r\,dr),\ u'(1) = 0 \text{ and } \lim_{r\to 0^+} \frac{u(r)}{\ln r} = 0\}.$$

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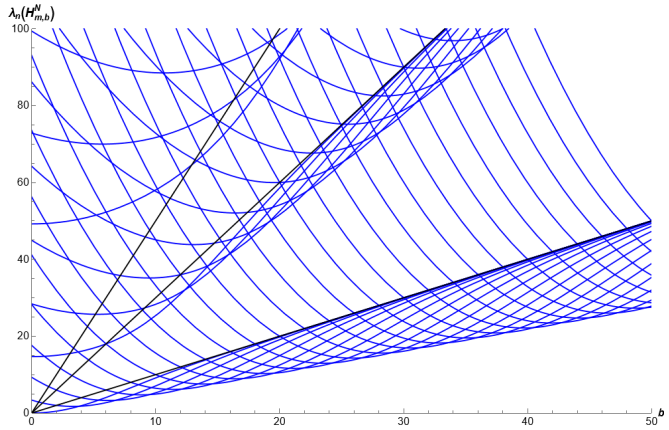

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Remark

We consider $m \geq 0$ since for $m < 0$

$$H^N_{m,b} = H^N_{-m,b} + 2|m|b.$$

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Theorem

Given $n \in \mathbb{N}$ and $m \in \mathbb{Z}$, if $b \to +\infty$, it holds that

$$\lambda_n(H^N_{m,b}) = (2n-1+|m|-m)b - e^{-\frac{b}{2}} \frac{b^{2n+m}}{(n-1)!(m+n-1)!2^{2(n-1)+m}} \left(1+\mathcal{O}(b^{-1})\right).$$

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Scheme of the proof: Trial state

$$u_{m,n}(r) := r^m L^m_{n-1}\left(\frac{br^2}{2}\right)\left(C_1 e^{\frac{b}{4}(1-r^2)} + C_2 e^{-\frac{b}{4}(1-r^2)}\right),$$

where C_1, C_2 are constants to determine such that $u_{m,n} \in \mathcal{D}(H^N_{m,b})$ and L^m_{n-1} are the associated Laguerre polynomials.

- Upper bound: Rayleigh–Ritz variational formula (min-max principle)
- Lower bound: Temple inequality

Main result and scheme of the proof

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Remark

Let $f(r) = r^m e^{-\frac{br^2}{4}} L^m_{n-1}(br^2/2)$, then $H_{m,b} f = (2n-1)bf$.

Boundary conditions

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Neumann:

$$u'_{m,n}(1) = 0 \Rightarrow C_2 = \frac{(\frac{b}{2} - m)L^m_{n-1}(\frac{b}{2}) - b(L^m_{n-1})'(\frac{b}{2})}{(\frac{b}{2} + m)L^m_{n-1}(\frac{b}{2}) + b(L^m_{n-1})'(\frac{b}{2})} C_1,$$

and for $b \to +\infty$, we have $C_2 = \big(1 + \mathcal{O}(b^{-1})\big)C_1$, so we can take $C_1 = 1$ and $C_2(b) = 1 + \mathcal{O}(b^{-1})$.

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and for $b \to +\infty$, we have $C_2 = \big(1 + \mathcal{O}(b^{-1})\big)C_1$, so we can take $C_1 = 1$ and $C_2(b) = 1 + \mathcal{O}(b^{-1})$.

Dirichlet:

$$u_{m,n}(1) = 0 \Rightarrow C_2 = -C_1$$

Remark

Robin boundary condition $u'_{m,n}(1) = \gamma u_{m,n}(1)$ with $\gamma \in \mathbb{R}$ will be analogous to the Neumann case for fixed parameter γ.

The Rayleigh–Ritz variational method

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By min-max principle

$$\lambda_n(H^N_{m,b}) = \min_{\substack{U\subset\mathcal{D}(H^N_{m,b})\\ \dim U=n}} \max_{u\in U\setminus\{0\}} \frac{\langle H^N_{m,b}u, u\rangle}{\langle u, u\rangle}$$

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- $\{u_{m,1},\dots,u_{m,n}\}$ spans a subspace of dimension n.

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- $\{u_{m,1},\dots,u_{m,n}\}$ spans a subspace of dimension n.

$$\lambda_n(H^N_{m,b}) \leq (2n-1)b - e^{-\frac{b}{2}} \frac{b^{2n+m}}{(n-1)!(m+n-1)!2^{2(n-1)+m}}(1+\mathcal{O}(b^{-1})).$$

The Temple inequality

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Theorem (Temple inequality)

Let $u \in \mathcal{D}(A) \setminus \{0\}$ *and suppose that there exists* $\mu, \nu \in \mathbb{R}$ *such that*

$$\lambda_{n-1}(A) \leq \mu < \frac{\langle Au, u\rangle}{\|u\|^2} < \nu \leq \lambda_{n+1}(A),$$

then

$$\frac{\nu\langle Au, u\rangle - \|Au\|^2}{\nu\|u\|^2 - \langle Au, u\rangle} \leq \lambda_n(A) \leq \frac{-\mu\langle Au, u\rangle + \|Au\|^2}{-\mu\|u\|^2 + \langle Au, u\rangle}.$$

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then

$$\frac{\nu\langle Au, u\rangle - \|Au\|^2}{\nu\|u\|^2 - \langle Au, u\rangle} \leq \lambda_n(A) \leq \frac{-\mu\langle Au, u\rangle + \|Au\|^2}{-\mu\|u\|^2 + \langle Au, u\rangle}.$$

For all $i \in \mathbb{N}$,

$$(\lambda_i(A) - \lambda_n(A))(\lambda_i(A) - \nu)\langle u, \varphi_i\rangle^2 \geq 0,$$

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then

$$\frac{\nu\langle Au, u\rangle - \|Au\|^2}{\nu\|u\|^2 - \langle Au, u\rangle} \leq \lambda_n(A) \leq \frac{-\mu\langle Au, u\rangle + \|Au\|^2}{-\mu\|u\|^2 + \langle Au, u\rangle}.$$

For all $i \in \mathbb{N}$,

$$(\lambda_i(A) - \lambda_n(A))(\lambda_i(A) - \nu)\langle u, \varphi_i\rangle^2 \geq 0,$$

$$\sum_i \lambda_i(A)^2\langle u, \varphi_i\rangle^2 - (\lambda_n(A)+\nu)\sum_i \lambda_i\langle u, \varphi_i\rangle^2 + \lambda_n(A)\nu\sum_i \langle u, \varphi_i\rangle^2 \geq 0,$$

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Theorem (Temple inequality)

Let $u \in \mathcal{D}(A) \setminus \{0\}$ and suppose that there exists $\mu, \nu \in \mathbb{R}$ such that

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For all $i \in \mathbb{N}$,

$$(\lambda_i(A) - \lambda_n(A))(\lambda_i(A) - \nu)\langle u, \varphi_i\rangle^2 \geq 0,$$

$$\|Au\|^2 - (\lambda_n(A) + \nu)\langle Au, u\rangle + \lambda_n(A)\nu\|u\|^2 \geq 0.$$

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Suppose that there are $\nu_{n+1}, \mu_{n-1} > 0$ and a trial state $u \in \mathcal{D}(H^N_{m,b}) \setminus \{0\}$ satisfying

$$\lambda_{n-1}(H^N_{m,b}) \leq \mu_{n-1} < \frac{\langle H^N_{m,b} u, u\rangle}{\|u\|^2} < \nu_{n+1} \leq \lambda_{n+1}(H^N_{m,b}),$$

then we get the following bound from below

$$\frac{\nu_{n+1}\langle H^N_{m,b} u, u\rangle - \langle H^N_{m,b} u, H^N_{m,b} u\rangle}{\nu_{n+1}\|u\|^2 - \langle H^N_{m,b} u, u\rangle} \leq \lambda_n(H^N_{m,b}),$$

for $n \in \mathbb{N}$.

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for $n \in \mathbb{N}$.
For μ_{n-1} we can use the upper bound we found before.
We need to find a rough lower bound ν_{n+1}.

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Proposition

For any integer $m \geq 0$, there exist $C, b_0 > 0$ such that, for $n = 1, 2, 3, \ldots$ and $b \geq b_0$, we have

$$\lambda_n(H^N_{m,b}) \geq (2n-1)b - C.$$

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$$\lambda_n(H^N_{m,b}) \geq (2n-1)b - C.$$

Idea of the proof: let $u \in \mathcal{D}(H^N_{m,b})$, consider a partition of unity $\{\chi_1, \chi_2\}$ with $\chi_k \in C^\infty(\mathbb{R}; [0,1])$,

$$\text{supp } \chi_1 \subset \left(-\infty, \frac{3}{4}\right), \qquad \text{supp } \chi_2 \subset \left(\frac{1}{2}, +\infty\right),$$

and $\chi_1^2 + \chi_2^2 = 1$. This relates $H^N_{m,b}$ with two self-adjoint operators with Dirichlet boundary condition at $r = \frac{3}{4}$ and $r = \frac{1}{2}$ respectively.

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- $\nu_{n+1} := (2n-1)b - C.$

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$$\frac{\nu_{n+1}\langle H^N_{m,b}u_{m,n}, u_{m,n}\rangle - \langle H^N_{m,b}u_{m,n}, H^N_{m,b}u_{m,n}\rangle}{\nu_{n+1}\|u_{m,n}\|^2 - \langle H^N_{m,b}u_{m,n}, u_{m,n}\rangle} \leq \lambda_n(H^N_{m,b}),$$

- $\nu_{n+1} = (2n-1)b - C$

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- $\nu_{n+1} = (2n-1)b - C$
- $\|u_{m,n}\|^2 = \frac{(m+n-1)!2^m}{b^{m+1}(n-1)!}e^{\frac{b}{2}} + \mathcal{O}(b^{2n-2})$

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$$\frac{\nu_{n+1}\langle H^N_{m,b}u_{m,n}, u_{m,n}\rangle - \langle H^N_{m,b}u_{m,n}, H^N_{m,b}u_{m,n}\rangle}{\nu_{n+1}\|u_{m,n}\|^2 - \langle H^N_{m,b}u_{m,n}, u_{m,n}\rangle} \leq \lambda_n(H^N_{m,b}),$$

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- $\langle H^N_{m,b}u_{m,n}, u_{m,n}\rangle = (2n-1)b\|u_{m,n}\|^2 - \frac{b^{2n-1}}{2^{2n-2}((n-1)!)^2} + \mathcal{O}(b^{2n-2}).$

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- $\langle H_{m,b}u_{m,n}, H_{m,b}u_{m,n}\rangle$ lower order terms

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$$\frac{\nu_{n+1}\langle H^N_{m,b}u_{m,n}, u_{m,n}\rangle - \langle H^N_{m,b}u_{m,n}, H^N_{m,b}u_{m,n}\rangle}{\nu_{n+1}\|u_{m,n}\|^2 - \langle H^N_{m,b}u_{m,n}, u_{m,n}\rangle} \leq \lambda_n(H^N_{m,b}),$$

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- $\langle H^N_{m,b}u_{m,n}, u_{m,n}\rangle = (2n-1)b\|u_{m,n}\|^2 - \frac{b^{2n-1}}{2^{2n-2}((n-1)!)^2} + \mathcal{O}(b^{2n-2})$.
- $\langle H_{m,b}u_{m,n}, H_{m,b}u_{m,n}\rangle$ lower order terms

Combining the terms above we get

$$(2n-1)b - e^{-\frac{b}{2}}\frac{b^{2n+m}}{(n-1)!(m+n-1)!2^{2(n-1)+m}}(1+\mathcal{O}(b^{-1})) \leq \lambda_n(H^N_{m,b}).$$

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- Eigenvalue asymptotics for the disc, as $b \to +\infty$

$$\lambda_1(H_b^N, D(0,1)) = \Theta_0 b - C_1\sqrt{b} + \mathcal{O}(1).$$

Moreover, $b \mapsto \lambda_1(H_b^N, D(0,1))$ is strictly increasing for large $b > 0$ *(Bauman, Phillips, Tang, Helffer, Morame, Fournais)*

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- Eigenvalue asymptotics for the complement of the disc, as $b \to +\infty$

$$\lambda_1(H_b^N, D(0,1)^C) = \Theta_0 b + C_1\sqrt{b} + \mathcal{O}(1).$$

(Helffer,Morame)

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(Helffer,Morame)

- Eigenvalue asymptotics for the complement of the disc, as $b \to 0^+$

$$\lambda_1(H_b^N, D(0,R)^C) = b - \frac{R^{2k}}{2^{k-1}(k-1)!} b^{k+1} + \mathcal{O}(b^{k+\frac{3}{2}}).$$

(Kachmar, Lotoreichik, Persson Sundqvist)

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Thank you for your attention.

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