Towards an Algebraic Approach to Gravitational Quantum Mechanics

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Abstract

Most approaches towards a quantum theory of gravitation indicate the existence of a minimal length scale of the order of the Planck length. Quantum mechanical models incorporating such an intrinsic length scale call for a deformation of Heisenberg's algebra resulting in a generalized uncertainty principle and constitute what is called gravitational quantum mechanics. Using an explicit position representation, the free time evolution of a Gaussian wave packet is investigated as well as the spectral properties of a particle bound by an external attractive potential. Here the cases of a box with finite width and infinite walls, and an attractive potential well of finite depth are considered.

Outline

The two fundamental discoveries of 20$^{\text{th}}$ century

General relativity: Einstein 1915

Quantum mechanics: Heisenberg (1925) and Schrödinger (1926)

Matvei Petrovich Bronstein in 1936
Quantentheorie schwacher Gravitationsfelder,
Physikalische Zeitschrift der Sowjetunion 9 (1936), 140–157.

Hartland S. Snyder in 1947
Quantized Space-Time,
Phys. Rev. 71 (1947) 38-41

$$[\hat{X}, \hat{P}] = \mathrm{i}\hbar \left(1 + \frac{\lambda^2}{\hbar^2}\,\hat{P}^2\right) , \qquad \lambda = \text{'natural unit of length'}$$

Quantum gravity: Minimal length $\qquad \lambda = \lambda_P \approx 10^{-33}$ cm

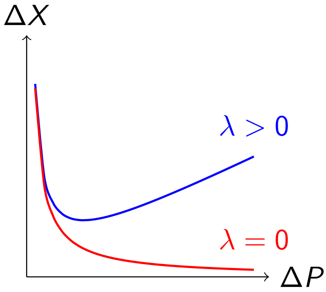
Generalised uncertainty principle (GUP)

Recall uncertainty relation:

$$\Delta X \Delta P \geq \frac{1}{2}\left\langle |[\hat{X},\hat{P}]| \right\rangle_\psi = \frac{\hbar}{2}\left(1 + \frac{\lambda^2}{\hbar^2}\langle \hat{P}^2\rangle_\psi\right)$$

With $\langle \hat{P}\rangle_\psi = 0$ (rest frame) we have the minimal uncertainty

$$\Delta X = \frac{1}{2}\left(\frac{\hbar}{\Delta P} + \frac{\lambda^2}{\hbar}\Delta P\right)$$



$\min \Delta X = \lambda \quad \text{at} \quad \Delta P = \hbar/\lambda$

Gravitational quantum mechanics (GQM)

Gravitational quantum mechanics

- Deformed Heisenberg algebra:

$$\boxed{[\hat{X},\hat{P}] = \mathrm{i}\hbar\left(1+\frac{\lambda^2}{\hbar^2}\,\hat{P}^2\right)}$$

with deformation parameter $\lambda > 0$ small [Length scale]

- Hamiltonian:

$$H = \frac{\hat{P}^2}{2M} + V(\hat{X})$$

for particle with mass $M > 0$

$\Longrightarrow$ Study effects for small non-vanishing λ

GUP momentum representation

From now on $\hbar = 1$: $[\hat{X}, \hat{P}] = \mathrm{i}\left(1 + \lambda^2 \hat{P}^2\right)$

Representation:

$$\hat{X} = \mathrm{i}\left(1 + \lambda^2 P^2\right) \frac{\partial}{\partial P}\,, \qquad \hat{P} = P$$

Hamiltonian:

$$H_P = \frac{P^2}{2M} + V\left(\mathrm{i}(1 + \lambda^2 P^2)\partial_P\right)$$

Scalar product:

$$\langle \Phi | \Psi \rangle = \int_{-\infty}^{\infty} \mathrm{d}P\, \frac{1}{1 + \lambda^2 P^2}\, \Phi^*(P)\Psi(P)$$

Here $\hat{P}$ is self-adjoint but $\hat{X}$ is only symmetric.
For details see: A. Kempf et al, PRD 52 (1996) 1108-1118.

Canonical momentum representation $[\hat{X},\hat{P}] = \mathrm{i}(1+\lambda^2\hat{P}^2)$

Recall: $\hat{x} = \mathrm{i}\partial_p\,,\qquad \hat{p} = p\,,\qquad$ with $\qquad [\hat{x},\hat{p}] = \mathrm{i}$

Representation:

$$\hat{X} = \mathrm{i}\partial_p\,,\qquad \hat{P} = \frac{1}{\lambda}\tan(\lambda p)$$

Hamiltonian:

$$H_p = \frac{1}{2M\lambda^2}\,\tan^2(\lambda p) + V\,(\mathrm{i}\partial_p)$$

Scalar product:

$$\langle\Phi|\Psi\rangle = \int_{-p_0}^{p_0}\mathrm{d}p\,\Phi^*(p)\Psi(p)\,,\qquad p_0 := \frac{\pi}{2\lambda}$$

Deformation function:

$$p = \varphi(P) := \frac{1}{\lambda}\arctan\left(\lambda P\right) \in [-p_0,p_0]$$

Note: $\qquad \varphi'(P) = \left(1+\lambda^2P^2\right)^{-1}$

GUP position representation $[\hat{X}, \hat{P}] = \mathrm{i}(1 + \lambda^2 \hat{P}^2)$

Representation:

$$\hat{X} = x \qquad \hat{P} = \frac{1}{\lambda}\tan(\lambda\hat{p}) = \frac{1}{\lambda}\tan(-\mathrm{i}\lambda\partial_x) = \frac{1}{\mathrm{i}\lambda}\tanh(\lambda\partial_x)$$

Hamiltonian:

$$H = -\frac{1}{2M\lambda^2}\,\tanh^2(\lambda\partial_x) + V(x)$$

Scalar product:

$$\langle\Phi|\Psi\rangle = \int_{-\infty}^{\infty} \mathrm{d}x\, \Phi^*(x)\Psi(x)$$

Approximate solutions: $\hat{P} = -\mathrm{i}\partial_x + \mathrm{i}\frac{\lambda^2}{3}\partial_x^3 + O(\lambda^4)$

For example, particle in a box and free particle by

K. Nozari and T. Azizi, Gen. Relativ. Gravit. 38 (2006) 735-742

Our aim is to obtain exact solutions!!!

GUP derivative

Recall deformation function:

$$\varphi(z) = \frac{1}{\lambda}\arctan(\lambda z)\,, \qquad \varphi^{-1}(z) = \frac{1}{\lambda}\tan(\lambda z)$$

GUP derivate:

$$D_\varphi := -\mathrm{i}\varphi^{-1}(\mathrm{i}\partial_x) = \frac{1}{\lambda}\tanh(\lambda\partial_x)$$

To be understood as power series: $a_n = \frac{2^{2n+2}}{(2n+2)!}(2^{2n+2}-1)B_{2n}$

$$D_\varphi = \partial_x - \frac{\lambda^2}{3}\partial_x^3 + \frac{2\lambda^4}{15}\partial_x^3 - \frac{17\lambda^6}{315}\partial_x^5 + \cdots = \sum_{n=0}^{\infty} a_n \lambda^{2n}\partial_x^{2n+1}$$

GUP Hamiltonian:

$$H = -\frac{1}{2M\lambda^2}\tanh^2(\lambda\partial_x) + V(x) = -\frac{1}{2M}D_\varphi^2 + V(x)$$

Eigenvalue problem: $H\,\Psi = E\,\Psi$

Properties of GUP derivative

GUP exponential: $\mathrm{e}^{ax} \quad \Longrightarrow \quad E_\varphi(a;x)$

$$E_\varphi(a;x) := \exp\left\{-\mathrm{i}\varphi(\mathrm{i}a)x\right\} \quad \text{with} \quad D_\varphi\, E_\varphi(a;x) = a\, E_\varphi(a;x)$$

GUP wave equation:

$$\left(D_\varphi^2 + \omega^2\right) y(x) = 0 \qquad \omega > 0\,,$$

has solutions

$$E_\varphi(\mathrm{i}\omega;x) = \mathrm{e}^{\mathrm{i}\varphi(\omega)x}\,, \qquad E_\varphi(-\mathrm{i}\omega;x) = \mathrm{e}^{-\mathrm{i}\varphi(\omega)x}\,,$$

$$C_\varphi(\mathrm{i}\omega;x) := \tfrac{1}{2}\left[\mathrm{e}^{\mathrm{i}\varphi(\omega)x} + \mathrm{e}^{-\mathrm{i}\varphi(\omega)x}\right] = \cos\left(\varphi(\omega)x\right)\,,$$

$$S_\varphi(\mathrm{i}\omega;x) := \tfrac{1}{2\mathrm{i}}\left[\mathrm{e}^{\mathrm{i}\varphi(\omega)x} - \mathrm{e}^{-\mathrm{i}\varphi(\omega)x}\right] = \sin\left(\varphi(\omega)x\right)$$

with wave length

$$\ell := \frac{2\pi}{\varphi(\omega)} = \frac{2\pi\lambda}{\arctan(\lambda\omega)} \geq 4\lambda \quad \text{as} \quad E_\varphi(\mathrm{i}\omega; x+\ell) = E_\varphi(\mathrm{i}\omega;x)$$

Free particle - plane waves:

GQM

$$-D_\varphi^2\, U(x) = P^2\, U(x)$$

$$U_P(x) = \sqrt{\frac{\varphi'(P)}{2\pi}}\, \mathrm{e}^{\mathrm{i}\varphi(P)x}$$

Orthonormal

$$\int_{-\infty}^{\infty} \mathrm{d}x\, U_P^*(x) U_{P'}(x) = \delta(P - P')$$

Almost Complete

$$\int_{-\infty}^{\infty} \mathrm{d}P\, U_P^*(x) U_P(x') = \frac{\sin\frac{\pi}{2\lambda}(x - x')}{\pi(x - x')}$$

Same as in A. Kempf et al

SQM

$$-\partial_x^2\, u(x) = p^2\, u(x)$$

$$u_p(x) = \sqrt{\frac{1}{2\pi}}\, \mathrm{e}^{\mathrm{i}px}$$

Orthonormal

$$\int_{-\infty}^{\infty} \mathrm{d}x\, u_p^*(x) u_{p'}(x) = \delta(p - p')$$

Complete

$$\int_{-\infty}^{\infty} \mathrm{d}p\, u_p^*(x) u_p(x') = \delta(x - x')$$

Gaussian wave packet

Gaussian: $\Phi(x) := \frac{1}{(2\pi\sigma^2)^{1/4}} \exp\left\{-\frac{x^2}{4\sigma^2}\right\}, \qquad ||\Phi||^2 = 1$

P-component:

$$\tilde{\Phi}(P) := \int_{-\infty}^{\infty} \mathrm{d}x\, \Phi(x) U_P^*(x) = \sqrt{\frac{\sqrt{2}\sigma}{\sqrt{\pi}(1+\lambda^2 P^2)}}\,\mathrm{e}^{-\sigma^2[\varphi(P)]^2}$$

Almost complete:

$$||\tilde{\Phi}||^2 = 1 - \mathrm{erfc}\left(\frac{\pi\sigma}{\lambda\sqrt{2}}\right) = 1 - \sqrt{\frac{2}{\pi^3}}\,\mathrm{e}^{-\frac{\pi^2\sigma^2}{2\lambda^2}}\,\frac{\lambda}{\sigma}\left[1 + O\left(\frac{\lambda^2}{\sigma^2}\right)\right]$$

Cross check:

$$\tilde{\tilde{\Phi}}(x) := \int_{-\infty}^{\infty} \mathrm{d}P\, \tilde{\Phi}(P) U_P(x)$$

$$\Longrightarrow \qquad |\Phi(x) - \tilde{\tilde{\Phi}}(x)| \leq \mathrm{erfc}\left(\frac{\pi\sigma}{\lambda\sqrt{2}}\right)$$

Gaussian as minimum uncertainty state

Recall $\Phi(x) := \frac{1}{(2\pi\sigma^2)^{1/4}} \mathrm{e}^{-\frac{x^2}{4\sigma^2}}$ with $(\Delta x)^2 = \sigma^2$

and $\tilde{\Phi}(P) = \left(\frac{2\sigma^2}{\pi}\right)^{1/4} \sqrt{\varphi'(P)}\, \mathrm{e}^{-\sigma^2\varphi^2(P)}$

With $P = \frac{1}{\lambda}\tan(\lambda\varphi) = \varphi + \frac{1}{3}\lambda^2\varphi^3 + O(\lambda^4)$

and using

$$\langle\varphi^{2m}\rangle_{\tilde{\Phi}} := \sqrt{\frac{2\sigma^2}{\pi}} \int_{-p_0}^{p_0} \mathrm{d}\varphi\, \varphi^{2m} \mathrm{e}^{-2\sigma^2\varphi^2} = \frac{(2m-1)!!}{2^{2m}\sigma^{2m}} + \mathrm{e}^{-\frac{\pi^2\sigma^2}{2\lambda^2}} O\left(\frac{\sigma^{2m-1}}{\lambda^{2m-1}}\right)$$

$$\Longrightarrow \qquad \langle P^2\rangle = \langle\varphi^2\rangle + \frac{2}{3}\lambda^2\langle\varphi^4\rangle + O(\lambda^4) = \frac{1}{4\sigma^2}\left(1 + \frac{\lambda^2}{2\sigma^2} + O(\lambda^4)\right)$$

$$\Delta x \Delta P = \frac{1}{2}\left(1 + \frac{\lambda^2}{4(\Delta x)^2} + O(\lambda^4)\right) = \frac{1}{2}\left(1 + \lambda^2\langle P^2\rangle + O(\lambda^4)\right)$$

Gaussian is minimum uncertainty state up to $O(\lambda^4)$

GUP Fourier transformation

Let

$$\tilde{\phi}(P) := \mathcal{F}[\phi] = \int_{-\infty}^{\infty} \mathrm{d}x\, \phi(x)\, U_P^*(x)$$

$$\tilde{\tilde{\phi}}(x) := \mathcal{F}^{-1}[\tilde{\phi}] = \int_{-\infty}^{\infty} \mathrm{d}P\, \tilde{\phi}(P)\, U_P(x)$$

For smooth wave functions ϕ: $\sigma_\phi^2 := \langle (x - \langle x \rangle_\phi)^2 \rangle_\phi \gg \lambda^2$

We expect $|\phi(x) - \tilde{\tilde{\phi}}(x)|$ to be very small

Approximate free time evolution of initial state ϕ_0:

$$\tilde{\phi}_t(P) = \exp\left\{ -\frac{\mathrm{i}P^2}{2M}\, t \right\} \tilde{\phi}_0(P)$$

Results in

$$\phi_t(x) \approx \tilde{\tilde{\phi}}_t(x) = \int_{-\infty}^{\infty} \mathrm{d}P\, \tilde{\phi}_t(P)\, U_P(x)$$

Gaussian time evolution

$$\tilde{\Phi}_t(P) = \left(\frac{2\sigma^2}{\pi}\right)^{1/4} \sqrt{\varphi'(P)}\, \mathrm{e}^{-\sigma^2\varphi^2(P)}\, \mathrm{e}^{-\mathrm{i}\frac{P^2}{2M}t}, \quad |\tilde{\Phi}_t(P)|^2 = |\tilde{\Phi}(P)|^2$$

$$\Longrightarrow \qquad \langle P^2\rangle_{\tilde{\Phi}_t} = \frac{1}{4\sigma^2}\left(1 + \frac{\lambda^2}{2\sigma^2} + O(\lambda^4)\right)$$

$$\tilde{\tilde{\Phi}}_t(x) := \int_{-\infty}^{\infty} \mathrm{d}P\, \tilde{\Phi}_t(P) U_P(x)$$

with

$$\mathrm{e}^{-\mathrm{i}\frac{P^2}{2M}t} = \mathrm{e}^{-\mathrm{i}\sigma^2\varphi^2\frac{t}{\tau}}\left(1 - \frac{\mathrm{i}}{3}\lambda^2\varphi^4\sigma^2\frac{t}{\tau} + O(\lambda^4)\right),$$

where $\qquad \tau := 2M\sigma^2$

Now let: $\qquad \sigma_t^2 := \sigma^2(1 + \mathrm{i}\frac{t}{\tau}) \qquad \Longrightarrow \qquad$ Gaussian integral

Gaussian time evolution

Result:

$$\tilde{\tilde{\Phi}}_t(x) = \Phi_t^{\lambda=0}(x)\left(1 - \mathrm{i}\frac{\lambda^2}{\sigma^2}\frac{t}{\tau}\left(\frac{1}{12}\frac{\sigma^8}{\sigma_t^8}\frac{x^4}{\sigma^4} - \frac{\sigma^6}{\sigma_t^6}\frac{x^2}{\sigma^2} + \frac{\sigma^4}{\sigma_t^4}\right)\right) + O\left(\frac{t^2}{\tau^2}\frac{\lambda^4}{\sigma^4}\right)$$

Is normalized up to and including $O(\frac{t^2}{\tau^2}) \Rightarrow \frac{\sigma^2}{\sigma_t^2} = 1 - \mathrm{i}\frac{t}{\tau} + O(\frac{t^2}{\tau^2})$

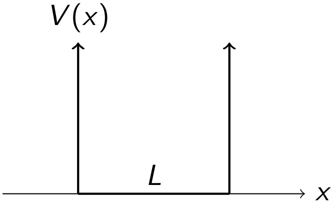
Time-dependent width:

$$\begin{aligned}\langle x^2\rangle_{\tilde{\tilde{\Phi}}_t} &:= \int \mathrm{d}x\, x^2 \left|\tilde{\tilde{\Phi}}_t(x)\right|^2 \\ &= \sigma^2\left(1 + \frac{t^2}{\tau^2} + 8\frac{\lambda^2}{\sigma^2}\frac{t^2}{\tau^2} + O\left(\frac{t^4}{\tau^4}\right) + O\left(\frac{t^2}{\tau^2}\frac{\lambda^4}{\sigma^4}\right)\right)\end{aligned}$$

Time-dependent uncertainty relation:

$$\boxed{(\Delta x)_t^2(\Delta P)_t^2 = \frac{1}{4}\left(1 + \frac{t^2}{\tau^2} + \frac{\lambda^2}{2\sigma^2} + \frac{17}{2}\frac{\lambda^2}{\sigma^2}\frac{t^2}{\tau^2} + \cdots\right)}$$

Particle in a box



GQM

$$\Psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi}{L} x$$

$$E_n(x) = \frac{1}{2M\lambda^2} \tan^2 \left(\frac{\lambda}{L} n\pi\right)$$

$$n = 1, 2, 3, \ldots, n_{max} < \frac{L}{2\lambda}$$

SQM

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi}{L} x$$

$$\varepsilon_n(x) = \frac{n^2\pi^2}{2ML^2}$$

$$n = 1, 2, 3, \ldots$$

Agrees to first order in λ^2 with K. Nozari and T. Azizi (but no n_{max})

Potentials with finite steps

Consider potential $V(x)$ with a finite jump at x_0 in GUP-SE

$$D_\varphi^2 \Psi(x) = 2M\,(V(x) - E)\,\Psi(x)$$

Consider "left-inverse of GUP derivative":

$$\int \mathrm{d}x \lim_{\lambda\to 0}\Big(D_\varphi f\Big)(x) = \int \mathrm{d}x\, f'(x) = f(x)$$

"GUP-integrate" $f = D_\varphi \Psi$ over interval $[x_0 - \varepsilon, x_0 + \varepsilon]$:

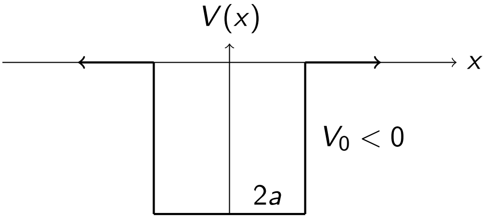
$$\Big(D_\varphi \Psi\Big)(x)\Big|_{x_0-\varepsilon}^{x_0+\varepsilon} = \int_{x_0-\varepsilon}^{x_0+\varepsilon} \mathrm{d}x\,\left[2M\,(V(x) - E)\,\Psi(x)\right]$$

Limit $\varepsilon \to 0$ implies boundary condition at x_0:

$$\Big(D_\varphi \Psi\Big)(x_0 - 0) = \Big(D_\varphi \Psi\Big)(x_0 + 0)$$

GUP-derivative expected to be continuous at x_0

Particle in a finite square well



Ansatz for bound states:

$$\Psi^{\pm}(x) = \pm\Psi(-x)\,, \qquad \kappa^2 = -2ME > 0\,, \qquad q^2 = 2M(E-V_0) > 0$$

$$\Psi^{+}(x) = \begin{cases} A\mathrm{e}^{\mathrm{i}\varphi(\mathrm{i}\kappa)\,|x|} & |x| > a \\ B\cos(\varphi(q)x) & |x| < a \end{cases}$$

$$\Psi^{-}(x) = \begin{cases} A\frac{x}{|x|}\mathrm{e}^{\mathrm{i}\varphi(\mathrm{i}\kappa)\,|x|} & |x| > a \\ C\sin(\varphi(q)x) & |x| < a \end{cases}\,,$$

Particle in a finite square well continued

Continuity of Ψ and $D_\varphi\Psi$ at $x = \pm a$ results in

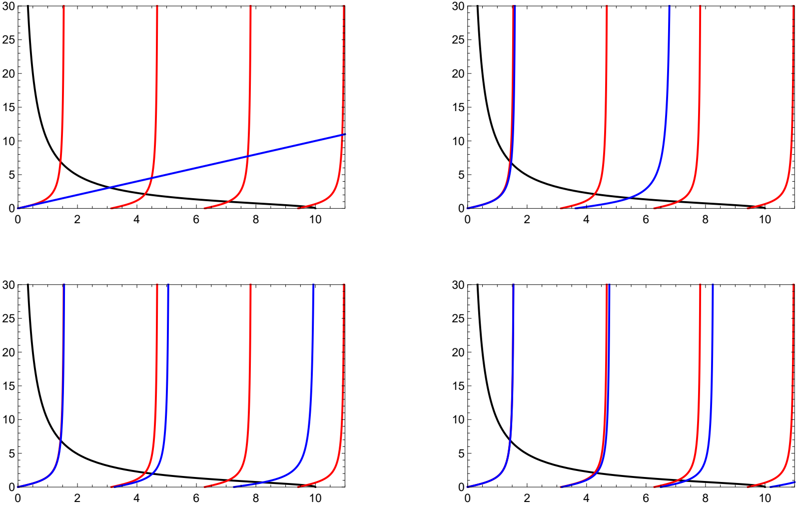
GQM	**SQM**	
$\kappa = q\tan\left(\varphi(q)a\right)$	$\kappa = q\tan(qa)$	sym.
$\kappa = -q\cot\left(\varphi(q)a\right)$	$\kappa = -q\cot(qa)$	anti-sym.

With $\gamma := a\sqrt{q^2+\kappa^2} = \sqrt{2Ma^2|V_0|}$ and $y := aq$

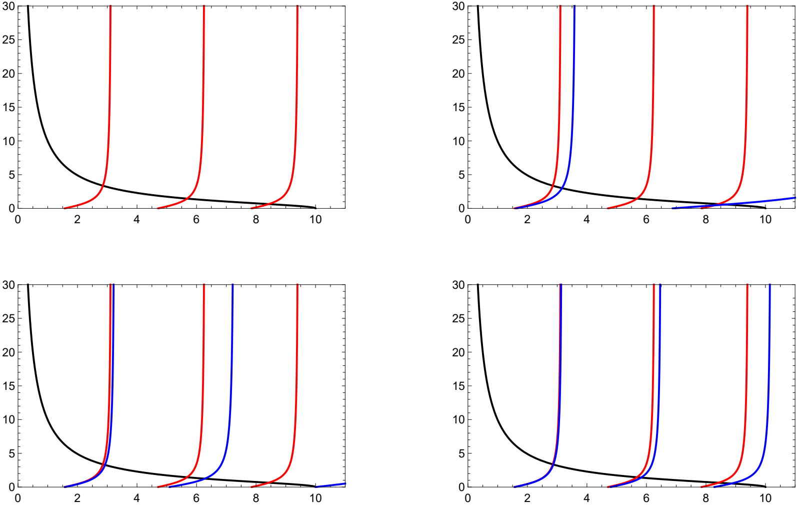
$$\frac{\sqrt{\gamma^2 - y^2}}{y} = \begin{cases} \tan\left(\frac{a}{\lambda}\arctan(\frac{\lambda}{a})y\right) & \text{symmetric solution} \\ -\cot\left(\frac{a}{\lambda}\arctan(\frac{\lambda}{a})y\right) & \text{anti-sym. solution} \end{cases}$$

Graphical solutions for $\gamma = 10$ on next slides.

Sym. solutions $a/\lambda = 1, 5, 10, 20$ for GQM and SQM



Anti-sym. sol. $a/\lambda = 1, 5, 10, 20$ for GQM and SQM



Discussion - Special cases

- For all $a > 0$ at least one bound state exists
- No excited states occur when $\gamma < \frac{a}{\lambda}\tan(\frac{a}{\lambda}\pi)$
- Case $a = \lambda$: Single bound state with

$$E_1 = V_0 + \frac{1}{4M\lambda^2}\left(\sqrt{1 - 8V_0 M\lambda^2} - 1\right)$$

- Case $a \to 0$ such that $\alpha := a|V_0| = const.$

$$V(x) \to 2\alpha\delta(x) \qquad \text{but} \qquad E_1 \approx -\frac{a\alpha\pi^2}{4\lambda^2} \to 0$$

Alternative GUP-delta function defined via GUP integral ???

Summary: Gravitational vs. Standard QM

GQM	**SQM**
$[\hat{X},\hat{P}] = \mathrm{i}\left(1+\lambda^2\hat{P}^2\right)$	$[\hat{x},\hat{p}] = \mathrm{i}$
$\Delta X \Delta P \geq \frac{1}{2}(1+\lambda^2\langle\hat{P}^2\rangle)$	$\Delta x \Delta p \geq \frac{1}{2}$
$H = \dfrac{\hat{P}^2}{2M} + V(\hat{X})$	$H = \dfrac{\hat{p}^2}{2M} + V(\hat{x})$
$\hat{P} = \frac{1}{\mathrm{i}}\frac{1}{\lambda}\tanh(\lambda\partial_x)$	$\hat{p} = \frac{1}{\mathrm{i}}\partial_x$
$\left[-\frac{1}{2M\lambda^2}\tanh^2(\lambda\partial_x) + V(x)\right]\psi = E\psi$	$\left[-\frac{1}{2M}\partial_x^2 + V(x)\right]\psi = E\psi$
Highly non-trivial diff. eq.	Well-studied diff. eq.
few exact solutions with many open questions	many exact solutions see any QM textbook

Prague's most famous writer: Franz Kafka 1883 - 1924