Modified Lane-Emden equation based on gravity with deviation

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Lane-Emden equation
gravity with a deviation
Modification of the Lane-Emden equation based on the gravity with a deviation
Applying of calculation for a hydrostatic system

Lane-Emden equation

A polytropic process is a thermodynamic process that obeys the following relation:

Pressure **Constant**

$$P\,V^{n} = C$$

Volume **Polytropic index**

Some specific values correspond to particular cases:

- **n=0 Isobaric process**
- **n=1 Isothermal process**
- **n=γ Isentropic process and γ is the ratio of heat capacity at constant pressure to heat capacity at constant volume.** $\gamma=\frac{C_p}{C_V}$
- **n=∞ Isochoric process**

For negative n, such process is not allowed, because of the second low thermodynamics.

- ✓ Lane-Emden equation is classical of mathematical physics.
- ✓ Lane –Emden equation was **introduced** in **1870 by Lane** and was **studied** in **1907 by Emden**[1].
- ✓ The Lane-Emden equation describes the density distribution inside a polytropic star in hydrostatic equilibrium.

For hydrostatic equilibrium consider a self-gravitating, spherical, metric fluid in hydrostatic equilibrium[2].

What is polytropic?

A spherical self-gravitating object in hydrostatic equilibrium that has polytropic equation of state $P = K\rho^{\gamma}$
Is a polytropic.

γ is adiabatic index and n $= \frac{1}{\gamma-1}$ is polytropic index.

$$\left\{ \begin{array}{l} P = K\rho^{\gamma} \\ P_c = K{\rho_c}^{\gamma} \end{array} \right. \Longrightarrow P = P_c\left(\frac{\rho}{\rho_c}\right)^{1+\frac{1}{n}}$$

For the central point

[1] J.H. Lane, The American Journal of Science and Arts 50, 57 (1870)
[2] R. Emden, Gaskugeln, Teubner, Berlin (1907)

$$dP = -\rho\, g\, dr \xrightarrow{g(r) = \frac{G\,m}{r^2}} dP = -\rho \frac{G\,m}{r^2}\, dr$$

And by differentiating again from above equation and considering

$$\begin{cases} \rho = \rho_c \theta^n \\ P = K {\rho_c}^{1+\frac{1}{n}} \theta^{n+1} \end{cases}$$

for a polytropic hydrostatic equilibrium [3]:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 K \rho_c^{\frac{1}{n}} (n+1) \frac{d\theta}{dr}\right) = -4\pi G \rho_c \theta^n$$

And by considering $\xi = \frac{r}{\alpha}$ where $\alpha = \frac{K\rho_c^{\frac{10}{n}-1}(n+1)}{4\pi G}$.

[3] Tooper, R. F. (1964). Astrophysical Journal, vol. 140, p. 434, 140, 434.

We have the **Lane-Emden equations**:

$$\frac{1}{\xi^2}\frac{d}{d\xi}(\xi^2\frac{d\theta}{d\xi}) + \theta^n = 0$$

We see that the above equation is similar to Poisson equation:

$$\nabla^2\phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{d\xi}\right) = 4\pi G\rho$$

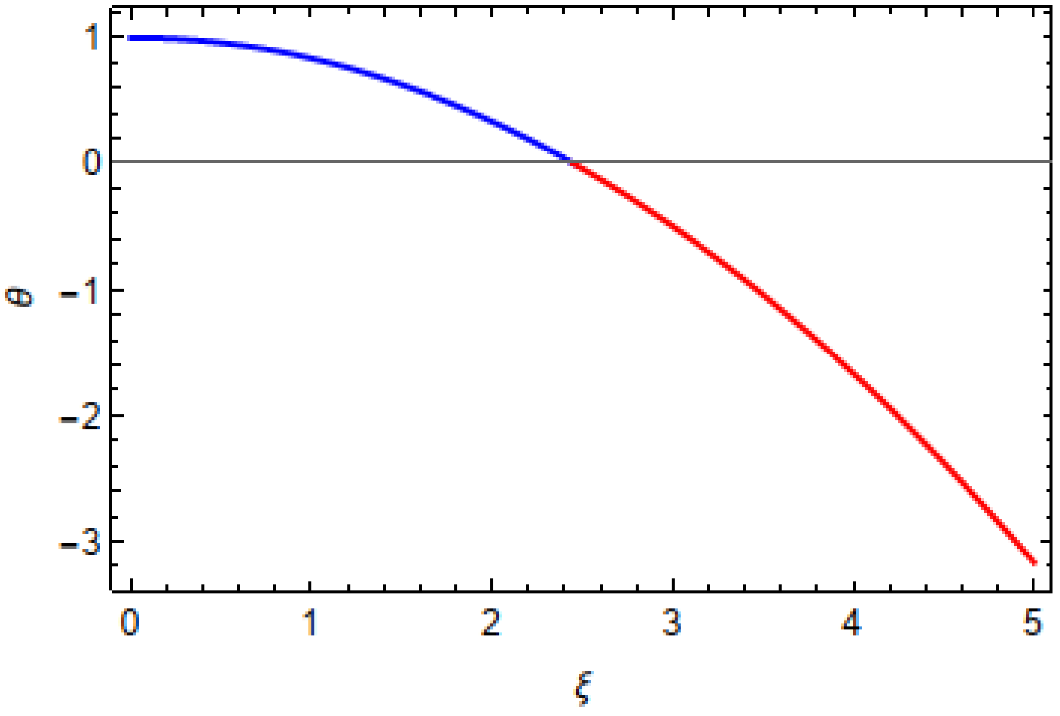
In the following table we have reported three solutions of the Lane-Emden equation :

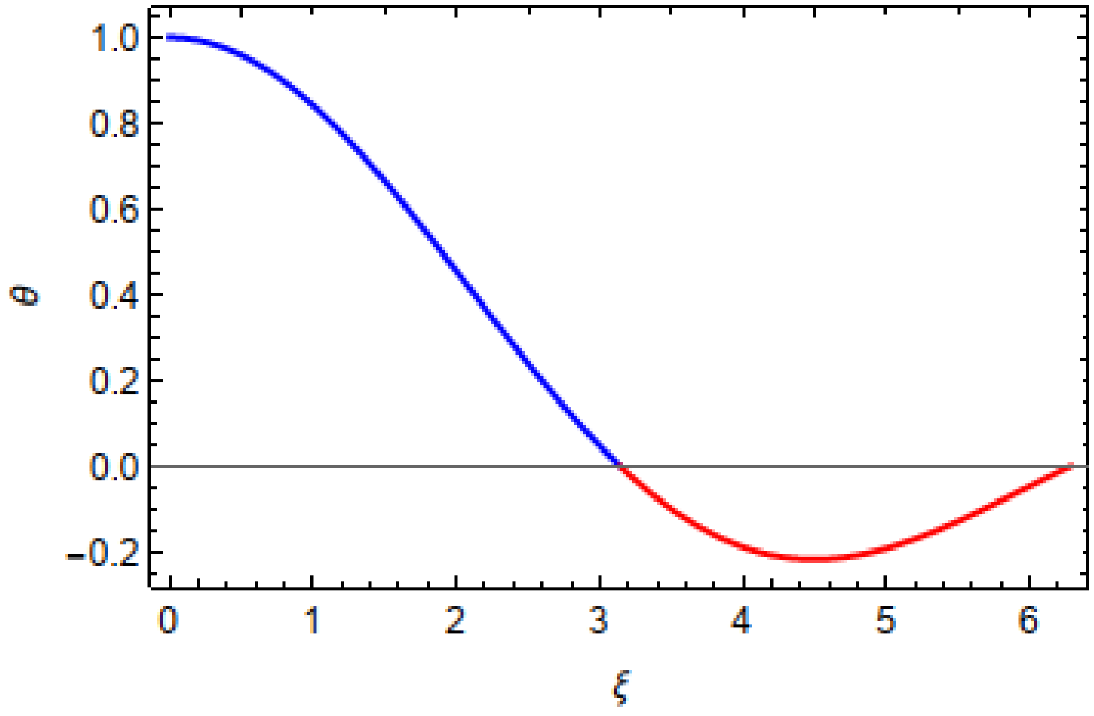
n	Boundary conditions	Function
0	$\theta(0)$=1,$\theta'(0)$=0	$\theta_0(\xi) = 1 - \frac{1}{6}\xi^2$
1	$\theta(0)$=1,$\theta'(0)$=0	$\theta_1(\xi) = \frac{Sin\xi}{\xi}$
5	$\theta(0)$=1,$\theta'(0)$=0	$\theta_5(\xi) = \frac{1}{\sqrt{1+\frac{\xi^2}{3}}}$

$$\rho = \rho_c \, \theta_n^n(\xi)$$

$$\theta_0(\xi) = 1 - \frac{1}{6}\xi^2$$

$$\theta_1(\xi) = \frac{Sin\xi}{\xi}$$





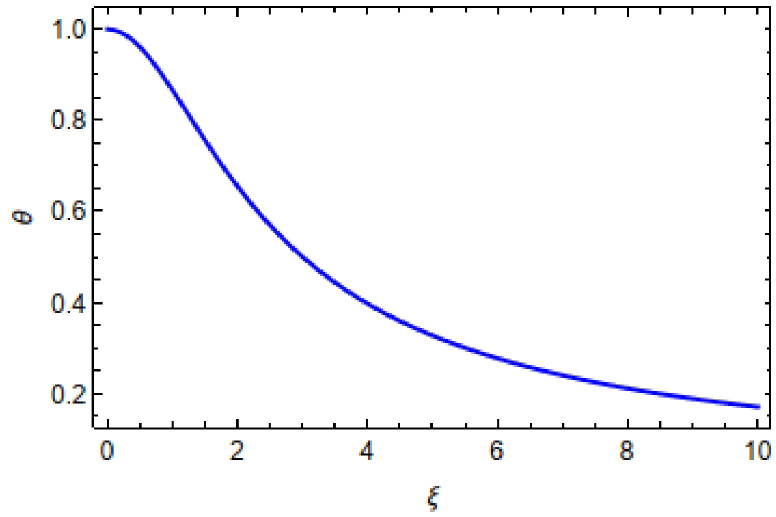
The blue area is the acceptable region and the red area is the unacceptable region.

In the Lane-Emden equation when **n=5**, by **Chandrasekhar** in **1932** are as follows [4]:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)+\theta^5=0$$

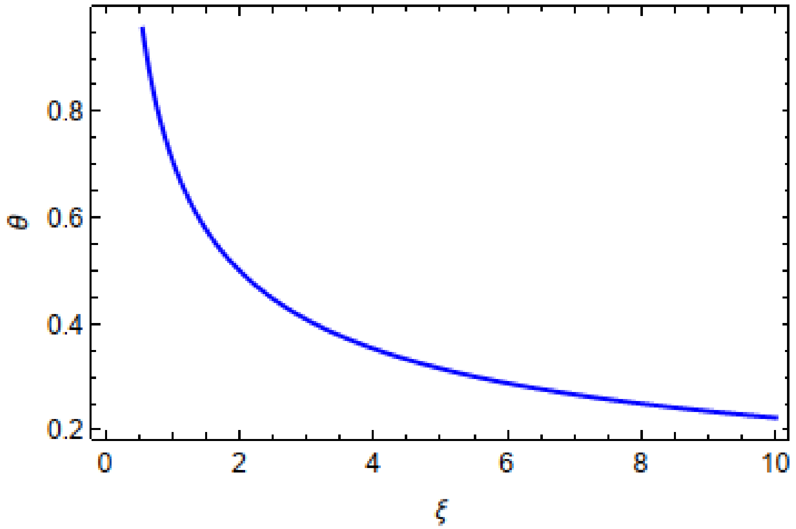
The first of them:

$$\theta_5(\xi)=\pm\frac{1}{\sqrt{1+\frac{\xi^2}{3}}}$$



The second:

$$\theta_5(\xi)=\pm\frac{1}{\sqrt{2\xi}}$$

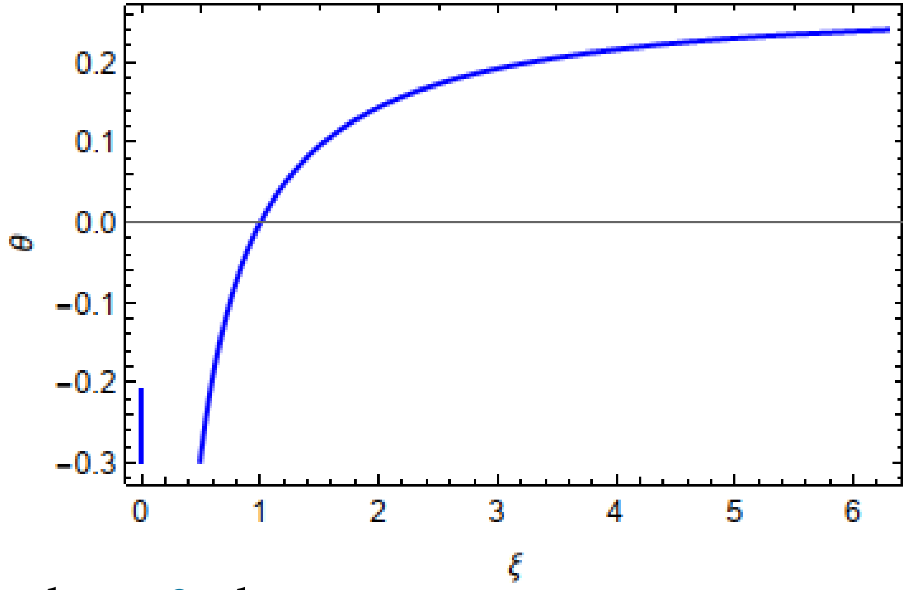


which is singular at $\xi = 0$.

[4] S. Chandrasekhar, University of Chicago Press, Chicago Illinois (1939)

Then, in 1962 **Srivastava** found another solution that can be written in a compact form, namely[5]:

$$\theta_5(\xi) = \pm \frac{Sin(ln\sqrt{\xi})}{\sqrt{3\xi - 2\xi\, Sin^2(ln\sqrt{\xi})}}$$



For all Lane– Emden equations with **n > 3**, that is:

$$\boxed{\theta\ \ (\xi) = (\frac{2(n-3)}{(n-1)^2\xi^2})^{1/(n-1)}}$$

[5] S. Srivastava, Astrophys. J. 136, 680 (1962)

A another solution given by **Patryk Mach** [6] for **n=5**, by substituting $\boldsymbol{\theta(\xi)} = \frac{\mathbf{z}}{\sqrt{\mathbf{2\xi}}}$ and $\mathbf{t} = \boldsymbol{ln\xi}$, the Lane-Emden equation becomes independent.

$$\frac{d^2z}{dt^2} = \frac{1}{4}z\left(1 - z^4\right)$$

and its standard integration yields:

$$\left(\frac{dz}{dt}\right)^2 = \frac{1}{12}\left(-z^6 + 3z^2 + C\right)$$

where C denotes an integration constant. Further analysis depends on the factorisation of the polynomial $\mathrm{W}(z) = -z^6 + 3z^2 + C$. In this equation z_1, z_2 and z_3 are $\mathrm{W}(z)$ real roots.

[6] Mach, P. (2012). Journal of mathematical physics, 53(6).

-2 < C < 0

$$\theta(\xi) = \pm\sqrt{\frac{z_1 z_2\, y^2}{2\xi\,(z_2 y^2 - (z_2 - z_1))}} \longrightarrow y = dc\left(\frac{1}{2}\sqrt{\frac{(z_1+z_3)z_2}{3}}\,Ln(B\xi), \sqrt{\frac{(z_2-z_1)z_3}{(z_1+z_3)z_2}}\right)$$

The **Jacobi dc** function is:

$$dc(U,m) = \frac{dn(U,m)}{cn(U,m)}$$

$dn(U,m) = \frac{d}{dU}a_m(U,m)$ and $a_m(U,m) = \varphi$ and U is incomplete elliptic integral of the first kind F:

$$U = F(\varphi,m) = \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - m\,sin^2\theta}}$$

$\varphi = a_m(U,m)$: Jacobi amplitude and elliptic cosine U is cn

$$cn(U,m) = \cos(\,a_m(U,m)) = \cos(\varphi)$$

Lane-Emden Solution for n=5

$0 < C < 2$

$$\theta(\xi) = \pm\sqrt{\frac{z_1 z_3\, y^2}{2\xi\,(z_1 y^2 + (z_1 + z_3)}} \longrightarrow y = sc\left(\frac{1}{2}\sqrt{\frac{(z_1 + z_3)z_2}{3}}Ln(B\xi), \sqrt{\frac{(z_2 - z_1)z_3}{(z_1 + z_3)z_2}}\right)$$

The elliptic sine is given by

$$sn(U,m) = \sin(\,a_m(U,m)) = \sin(\varphi)$$

and the delta amplitude

$$dn(U,m) = \frac{d}{dU}a_m(U,m)$$

and the **Jacobi sc** function is the ratio of the Jacobi elliptic sine function to the Jacobi elliptic function:

$$sc(U,m) = \frac{sn(U,m)}{cn(U,m)}$$

C > 0

$$\theta(\xi) = \pm \frac{\sqrt{C}}{\sqrt{2\xi}\sqrt{\wp\left(\frac{Ln(B\xi)}{(2\sqrt{3})}; 12{,}4(C^2-2)\right) - 1}}$$

The Weierstrass elliptic function:

$$\wp(\mathrm{z}, \omega_1, \omega_2) = \wp(\mathrm{z}) = \frac{1}{z^2} + \sum_{\{m,n\}\neq\{0,0\}} \left(\frac{1}{(z + 2\,m\,\omega_1 + 2\,n\,\omega_2)^2} - \frac{1}{(2\,m\,\omega_1 + 2\,n\,\omega_2)^2} \right)$$

This function is doubly periodic with fundamental periods equal to $2\omega_1$ and $2\omega_2$

$$\wp(\mathrm{z} + 2\omega_1) = \wp(\mathrm{z}), \qquad \wp(\mathrm{z} + 2\omega_2) = \wp(\mathrm{z})$$

In **1687**, Newton hypothesized the inverse-square law (ISL) of universal gravitation, where the gravitational force F acting on a particle with mass m located at r, due to the presence of a mass m’ located at origin was given by

$$\mathbf{F} = -G\frac{mm'\mathbf{r}}{r^3} = -G\frac{mm'}{r^2}\hat{r}$$

In **1894**, **Hall** applied **Bertrand's formula** a for the orbit of Mercury and proposed the gravitational force with deviation in the form [7]:

$$|\mathbf{F}| = G\frac{mm'}{|\mathbf{r}|^{2+\sigma}}$$

Hall determined the value of σ as σ = 0:00000016 to explain the discrepancy of 43 seconds of arc per century [8].

[7]A.Hall, the astronomical journal 319, 49 (1894).
[8]S. Plimpton, W. Lawton. Phys. Rev .(1936)

We assume that the **gravitational field** at r due to a **point mass** m located at the origin is given by:

$$\mathbf{g} = -\frac{Gm\mathbf{r}}{r^{3+\sigma}}, \qquad |\mathbf{g}| = g(r) = \frac{Gm}{r^{2+\sigma}},$$

The gravitational potential is defined by

$$\mathbf{g} = -\nabla V, \longrightarrow V = -\frac{Gm}{(1+\sigma)r^{1+\sigma}}$$

The Lane-Emden equation is a dimensionless form of Poisson's equation for the gravitational potential of a Newtonian self-gravitating, spherically symmetric, polytropic fluid [9].

In case of the deformed gravity, the Lane-Emden equation should also be modified. Let us consider a self-gravitating, spherically symmetric fluid in hydrostatic equilibrium, whose center is located at the origin [10].

[9]J. Lane, American Journal of Science. 2, 57 (1870).
[10]G. Panotopoulos and I. Lopes, International Journal of Modern Physics D, 29, 08 (2020).

The **gravitational field** at r is given by:

$$g(r) = \frac{GM(r)}{r^{2+\sigma}}, \qquad \text{where} \qquad M(r) = \int_0^r 4\pi\rho(r')r'^2 dr',$$

Now let us consider a star which is at hydrostatic equilibrium. The equation of motion for a tar radius is then given by:

$$\frac{dP(r)}{dr} = -\frac{G\rho(r)M(r)}{r^{2+\sigma}}$$

Of course the ordinary Lane-Emden equation can be obtained from the generalized **Tolman-Oppenheimer-Volkoff** equation when c goes to infinity, where c is speed of light [11,12].

$$\left.\begin{aligned} P(r) &= K\rho(r)^\gamma, \\ \gamma &= \frac{n+1}{n}, \end{aligned}\right\} \qquad \frac{n+1}{n}\frac{K}{r^2}\frac{d}{dr}\left(r^{2+\sigma}\rho^{\frac{1-n}{n}}\frac{d\rho}{dr}\right) = -4\pi G\rho.$$

[11]Makino, T. (2015). On the Tolman-Oppenheimer-Volkoff-de Sitter equation.
[12]R. Kippenhahn, A. Weigert and A. Weiss, Springer-verlag (Vol. 192) (1990).

We consider a star with radius r, mass M, and uniform density ρ. We consider mass $M(r) = \frac{4}{3}\pi r^3 \rho_c$, where ρ_c is central density. Using modified gravity, the potential energy due to gravity is calculated as follows:

$$U_g = \int_0^R \frac{GM(r)(4\pi r^2 \rho dr)}{(1+\sigma) r^{1+\sigma}}$$

if $\rho = \rho_c$, we have:

$$U_g = \Big(\frac{3}{(1+\sigma)(5-\sigma)}\Big)\Big(\frac{GM(R)^2}{R^{1+\sigma}}\Big)$$

The pressure due to gravity is calculated from the following equation:

$$P_g = -\partial_V U = \frac{1}{5-\sigma}\Big(GM(R)^2\Big)\Big((\frac{4\pi}{3})^{\frac{1}{3}(1+\sigma)} V^{\frac{-1}{3}(4+\sigma)}\Big)$$

For the deformed Lane-Emden equation, we consider the density as $\rho = \rho_c D_n^n(\xi)$. As a result, the deformed Lane-Emden equation is rewritten as follows:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{dD_n(\xi)}{d\xi}\right) + D_n^n(\xi) = 0$$

Case of n = 0

The modified Lane-Emden equation reads:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^{2+\sigma}\frac{dD_0(\xi)}{d\xi}\right) = -1.$$

By integrating :

$$\xi^{2+\sigma}\frac{dD_0(\xi)}{d\xi} = -\frac{\xi^3}{3} + c_1,$$

And finally we have:

$$D_0(\xi) = 1 - \frac{\xi^{2-\sigma}}{3(2-\sigma)}.$$

Case of n = 1

The modified Lane-Emden equation reads:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^{2+\sigma}\frac{dD_1(\xi)}{d\xi}\right) = -D_1(\xi)$$

By integrating :

$$D_1(\xi) = c_1\xi^{-\frac{1+\sigma}{2}} J_{\frac{1+\sigma}{2-\sigma}}\left(\frac{2}{2-\sigma}\xi^{1-\frac{\sigma}{2}}\right) + c_2\xi^{-\frac{1+\sigma}{2}} J_{-\frac{1+\sigma}{2-\sigma}}\left(\frac{2}{2-\sigma}\xi^{1-\frac{\sigma}{2}}\right)$$

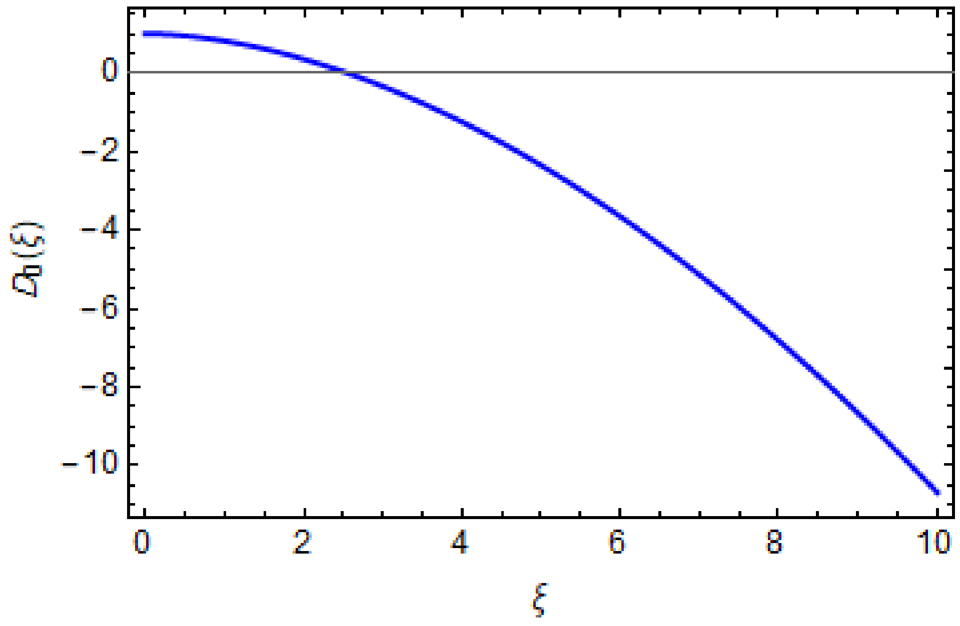
And finally we have:

$$D_1(\xi) = \Gamma\left(\frac{3}{2-\sigma}\right)(2-\sigma)^{\frac{1+\sigma}{2-\sigma}}\xi^{-\frac{1+\sigma}{2}} J_{\frac{1+\sigma}{2-\sigma}}\left(\frac{2}{2-\sigma}\xi^{1-\frac{\sigma}{2}}\right)$$

When σ goes to zero $D_n(\xi) \rightarrow \theta_n(\xi)$.

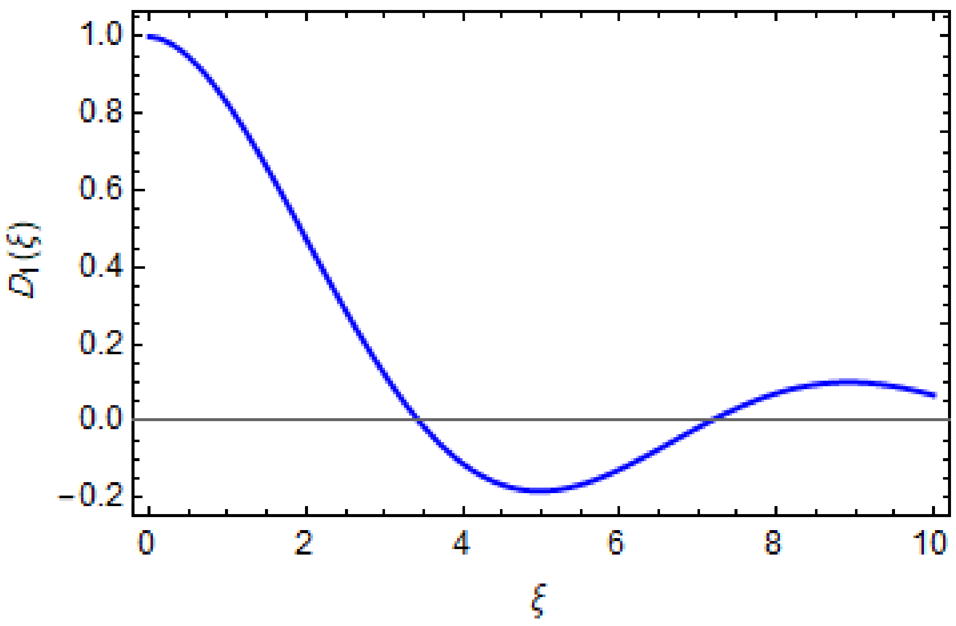
Case of n = 0

$$D_0(\xi) = 1 - \frac{\xi^{2-\sigma}}{3(2-\sigma)}.$$



Case of n = 1

$$D_1(\xi) = \Gamma\left(\frac{3}{2-\sigma}\right)(2-\sigma)^{\frac{1+\sigma}{2-\sigma}}\xi^{-\frac{1+\sigma}{2}}J_{\frac{1+\sigma}{2-\sigma}}\left(\frac{2}{2-\sigma}\xi^{1-\frac{\sigma}{2}}\right)$$



Case of n = 0

The potential due to gravity, by considering $\xi = \frac{r}{R}$,is obtained from the following equations:

$$U_{LE}^{n=0} = \int_0^R \frac{GM(r)(4\pi r^2)\rho_c D_0(\xi)dr}{(1+\sigma)r^{1+\sigma}} = \left(\frac{11+6\sigma}{4+4\sigma-\sigma^2-\sigma^3}\right)\left(\frac{GM(R)^2}{2R^{1+\sigma}}\right)$$

Using the following equation for the pressure due to modified gravity, we have:

$$P_{LE}^{n=0} = -\partial_V U_{LE}^{n=0} = \left(\frac{11+6\sigma}{4-\sigma^2}\right)\left(\frac{GM(R)^2}{8\pi R^{\sigma+4}}\right)$$

Case of n = 1

$$U_{LE}^{n=1} = \int_0^R \frac{GM(r)(4\pi r^2)\rho_c\left(\Gamma\left(\frac{3}{2-\sigma}\right)(2-\sigma)^{\frac{1+\sigma}{2-\sigma}}\xi^{-\frac{1+\sigma}{2}}J_{\frac{1+\sigma}{2-\sigma}}\left(\frac{2}{2-\sigma}\xi^{1-\frac{\sigma}{2}}\right)\right)dr}{(1+\sigma)r^{1+\sigma}}$$

Since the above integral cannot be solved parametrically, we have solved this integral for two different values of sigma and used the results to calculate the radius.

We intend to calculate the Fermi energy of the particle in the 3D box for the q-deformed formalism.

Using the Tsallis [13] distribution obtained from the inverse of the q-logarithmic function in Ref.[14] we have:

Deformed exponential function

$$e_q(x) = (1 + (1 - q)x)^{\frac{1}{1-q}}$$

Deformed parameter

Position

[13] Tsallis, C. (1988). Journal of statistical physics, 52, 479-487.
[14] Chung, W. S., and Hassanabadi, H. (2020). Modern Physics Letters A, 35(11), 2050074.

- **The q-deformed Schrödinger equation based on the q-map: one dimensional case**

WS Chung, SH Dong, H Hassanabadi The European Physical Journal Plus 139 (3), 1-15

- **Parity-deformed su(2) and so(3) Algebras: a Basis for Quantum Optics and Quantum Communications Applications**

  WS Chung, H Hassanabadi, LM Nieto, S Zarrinkamar, arXiv preprint arXiv:2407.12157

- **Modified Lane-Emden Equation and Modified Jeans' Instability Based Gravity with Deviation**

WS Chung, F Kafikang, H Hassanabadi, International Journal of Theoretical Physics 63 (7), 167

- **Two types of q-Gaussian distributions used to study the diffusion in a finite region**

Won Sang Chung, Luis M. Nieto, Soroush Zare and Hassan Hassanabadi

- **The Dunkl oscillator on a space of nonconstant curvature: An exactly solvable quantum model with reflections**

A Ballesteros, A Najafizade, H Panahi, H Hassanabadi, SH Dong, Annals of Physics 460, 169543

- **Radius of the white dwarf according to Fermi energy in a -deformed framework**

F Kafikang, H Hassanabadi, WS Chung, The European Physical Journal Plus 138 (6), 1-9

- **DKP Equation in the q-deformed Quantum Mechanics**

H Sobhani, H Hassanabadi, WS Chung, Few-Body Systems 64 (2), 18

We intend to calculate the Fermi energy of the particle in the one box for the q-deformed formalism.

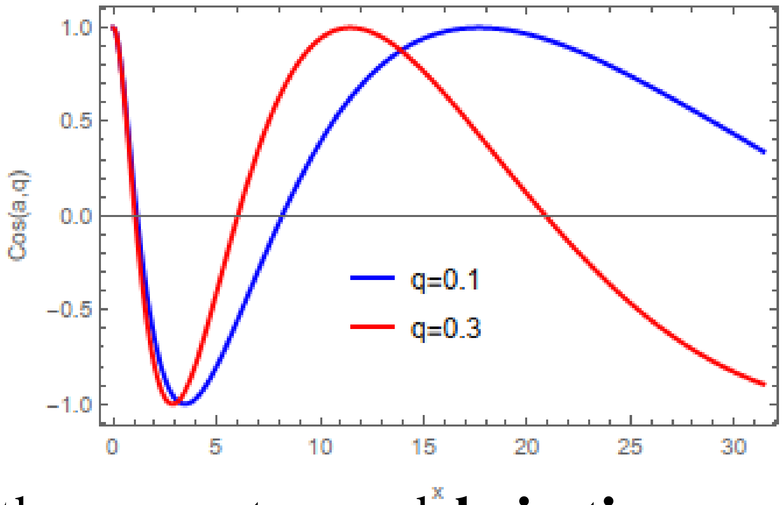
To calculate the trigonometric functions in this formalism, we first define the hyperbolic functions **q-deformed** as follows:

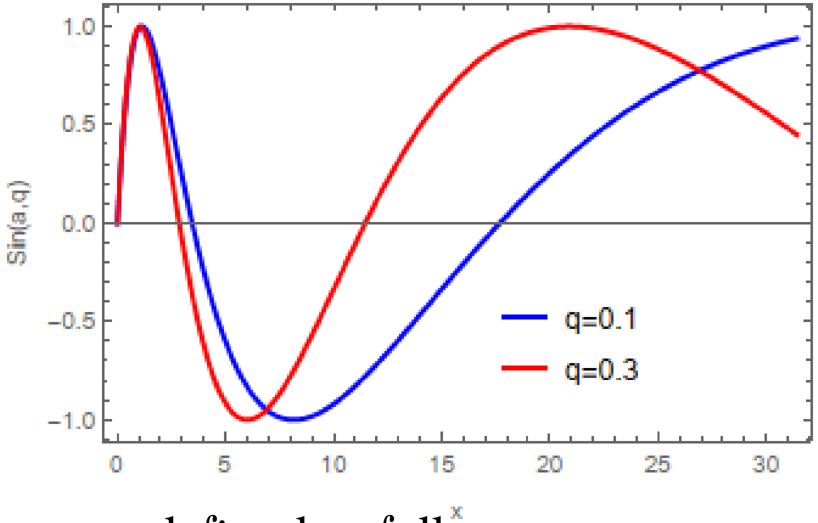
$$e_q(x) = \cosh_q(x) + \sinh_q(x),$$

$$Cosh_q(a\odot x) = cosh\left(\frac{a}{1-q}Ln(1+(1-q)x)\right) \qquad Sinh_q(a\odot x) = sinh\left(\frac{a}{1-q}Ln(1+(1-q)x)\right)$$

which is **q-Euler** formula, where **q-sine and q-cosine** function are defined by:

$$C_q(a\odot x) = \cos\left(\frac{a}{1-q}\ln(1+(1-q)x)\right) \qquad S_q(a\odot x) = \sin\left(\frac{a}{1-q}\ln(1+(1-q)x)\right)$$





Also, the **momentum** and **derivative operators** in this formalism are defined as follows:

$$\hat{p} = \frac{\hbar}{i}D_x, \qquad D_x = (1+qx)\partial_x$$

We consider a star, spinless particles of mass m are located in a box of length L, where the potential is defined as follows:

$$V(x) = \begin{cases} 0 & 0 < x < L \\ \infty & \text{elsewhere.} \end{cases}$$

The Schrödinger equation in three dimensions in the presence of q-deformed is rewritten as follows:

$$-\frac{\hbar^2}{2m_e} D_x^2\, u(x) = E\, u(x)$$

Where:

$$D_x^2 = q\, \partial_x + (1 + qx)\partial_x^2$$

wave functions has the following form:

$$u_n(x) = \left(\sqrt{\frac{2(1-q)}{Ln(1+(1-q)L)}}\right) \sin\left(n_x\, \pi \frac{Ln(1+(1-q)x)}{Ln(1+(1-q)L)}\right)$$

where n_x is energy level, as a result, the energy is calculated as follows:

$$E_{n_x} = \left(\frac{(1-q)^2 \hbar^2 \pi^2 n_x^2}{2\, m_e \big(Ln(1+(1-q)L)\big)^2}\right)$$

In white dwarfs, the pressure due to gravity is equal to the pressure due to degeneration, which can be used to calculate the radius of the white dwarf.

Fermi energy in a three-dimensional box problem in the presence of q-deformed as follows:

$$E_F = \left(\frac{\ln(2-q_0)}{1-q_0}\right)^{-2}\left(\frac{\hbar^2\pi^2}{2m_e}\right)\left(\frac{3N}{\pi V}\right)^{2/3}$$

The internal energy is calculated from the following equation:

$$U_d^q = \frac{3}{5}\left(\frac{\ln(2-q_0)}{1-q_0}\right)^{-2}\left(\frac{\hbar^2\pi^2}{2m_e}\right)\left(\frac{3N}{\pi V}\right)^{2/3}N,$$

The pressure due to degeneration is calculated from the following equation:

$$P_d^q = -\partial_V U_d = \frac{\left(\frac{\ln(2-q_0)}{1-q_0}\right)^{-2}}{5}\left(\frac{\hbar^2\pi^2}{m_e}\right)\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{N}{V}\right)^{5/3}$$

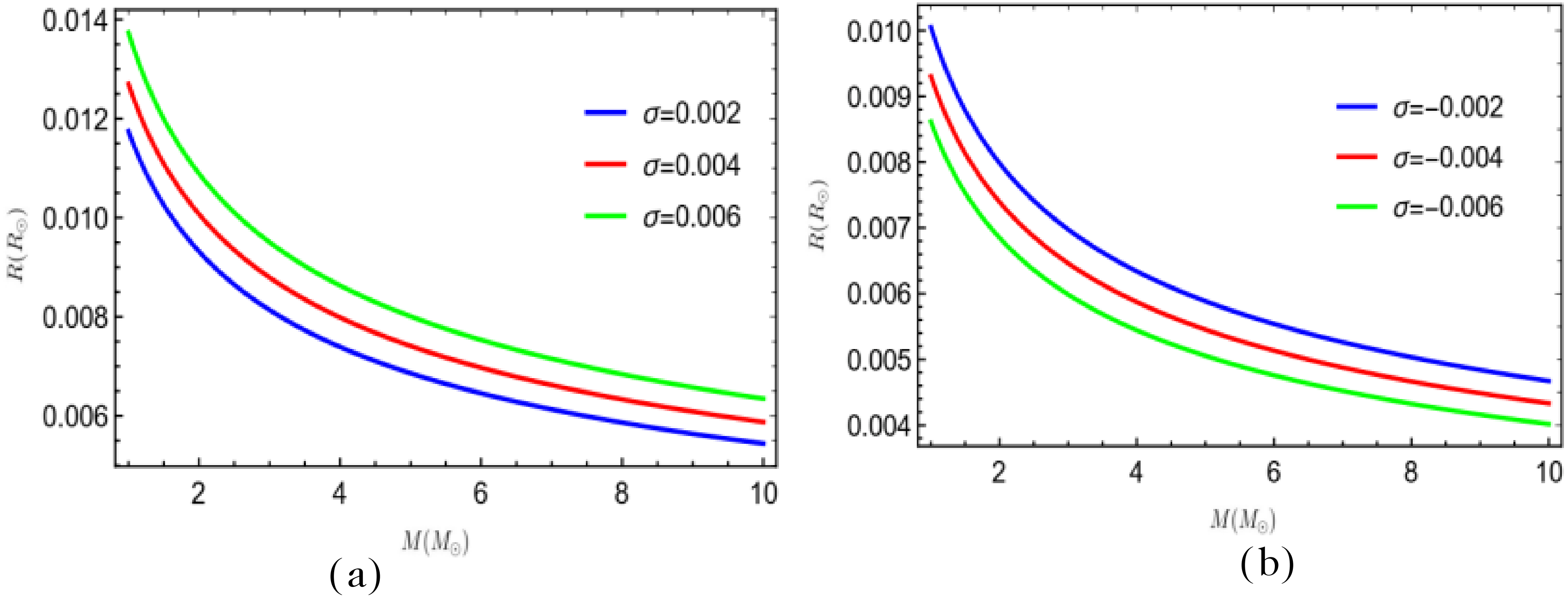
In white dwarfs, we have:

$$P_d = P_g$$

If n = 0, the radius of the white dwarf is calculated from the following equation:

$$R_{LE-q}^{n=0} = \left(\frac{2187\pi^2}{20^3}\right)^{\frac{1}{3-3\sigma}} \left(\frac{Gm_e m_p^{\frac{5}{3}} \left(\frac{\ln(2-q_0)}{1-q_0}\right)^2 \mu^{\frac{5}{3}}}{(4-\sigma^2)\hbar^2}\right)^{\frac{1}{\sigma-1}} M^{-\frac{1}{3}\left(\frac{1}{1-\sigma}\right)}$$

If n = 1 and $q_0 = 0.4$:



Plot of the $R_{LE}^{n=1}$ versus M (a) $\sigma > 0$ b) $\sigma < 0$.

Thanks for your attention