Quadratic integrable systems in magnetic fields: the extended cylindrical and spherical cases

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Motivation

Integrable and superintegrable systems are distinguished by their symmetry, which enables to solve their equations of motion exactly, in quadratures or even algebraically, respectively. Due to these properties they serve as a basis for perturbative models of more complex situations.

However, most integrable and superintegrable system were studied in the absence of magnetic field. Although this consideration is well justified on the mathematical grounds, as it makes the analysis considerably easier, there are many physical situations when magnetic field is present, e.g. in plasma physics. We therefore deem it necessary to extend the current knowledge in this direction.

Contents

Quadratic integrability and superintegrability, magnetic field

We consider integrable and superintegrable systems in three spatial dimensions.

Integrability

A classical Hamiltonian system with n degrees of freedom is integrable if it admits n functionally independent integrals of motion in involution.

Superintegrability

A classical Hamiltonian system with n degrees of freedom is polynomially superintegrable if it admits $n+k$ functionally independent integrals of motion (where $k \leq n-1$), that are polynomial in the momenta and out of which n are in involution.

In 3D: 4 integrals = minimal superintegrability, 5 integrals = maximal superintegrability.

Magnetic field and canonical momenta

The vector potential is represented by a 1-form

$$A(\vec{x}) = A_1(\vec{x})\mathrm{d}x_1 + A_2(\vec{x})\mathrm{d}x_2 + A_3(\vec{x})\mathrm{d}x_3$$

and the magnetic field 2-form

$$B = \mathrm{d}A = \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right)\mathrm{d}x_2 \wedge \mathrm{d}x_3 + \text{cycl.}$$

We stress that the formula holds in curvilinear coordinates.
In our choice of units the charge of the particle is -1 and we define the momenta covariant with respect to a time-independent gauge transformation $A'(\vec{x}) = A(\vec{x}) + \mathrm{d}\chi(\vec{x}),\ W'(\vec{x}) = W(\vec{x})$ as

$$p_i^A = p_i + A_i = mv_i.$$

The Hamiltonian system and integrals

We assume the 3D Euclidean space, therefore the Hamiltonian of the system after scaling the mass to 1 reads

$$H(\vec{x},\vec{p}) = \frac{1}{2}\sum_{j=1}^{3}\left(p_i^A\right)^2 + W(\vec{x}).$$

Note that the Hamiltonian contains first order terms in momenta. We assume a second order integral,

$$X = \sum_{j=1}^{3} h^j(\vec{x})p_j^A p_j^A + \sum_{j,k,l=1}^{3}\frac{1}{2}|\epsilon_{jkl}|n^j(\vec{x})p_k^A p_l^A + \sum_{j=1}^{3} s^j(\vec{x})p_j^A + m(\vec{x}).$$

Reduced integral

The condition for the integral X,

$$\{H, X\} = 0,$$

separates into terms of order $3, 2, 1$ and 0 in the momenta. The third order ones do not depend on $\vec{B}$ and can thus be easily solved. We get the following form of the integral with 20 constants α_{ij}, β_{ij} and γ_{ij}. Here we use the covariant angular momenta $\ell_i^A = \epsilon_{ijk} x_j p_k^A$.

$$X = \sum_{i,j:\ i \leq j} \alpha_{ij} \ell_i^A \ell_j^A + \sum_{i,j} \beta_{ij} p_i^A \ell_j^A + \sum_{i,j:\ i \leq j} \gamma_{ij} p_i^A p_j^A + \\ + s_1(\boldsymbol{x}) p_1^A + s_2(\boldsymbol{x}) p_2^A + s_3(\boldsymbol{x}) p_3^A + m(\boldsymbol{x}).$$

The lower order ones imply 10 linear PDEs on the functions $\vec{s}, m, \vec{B}, W$ which also depend on the constants α_{ij}, β_{ij} and γ_{ij}.

Quadratic (super)integrability without magnetic field

No magnetic field:

- Completely classified [Makarov et al. 1967].
- The highest order terms of these integrals are determined by a 1:1 correspondence with orthogonal separation of variables of Hamilton–Jacobi equation.
- Superintegrable systems are multiseparable, i.e. separate in several orthogonal coordinate systems.

Quadratic (super)integrability with magnetic field

With magnetic field

- Only some separable cases were explored, far short of classification (especially superintegrable).
- The 1:1 correspondence is broken: Separability is much more constraining, implies at least one first order integral.
- The highest order terms of the integrals may no longer be constrained to the separable cases. Using the highest order determining equations, the forms of these terms were explored in [Marchesiello and Šnobl 2022]. A particular integrable system was found.
- To classify integrable systems, the lower order determining equations must be imposed to confirm that the systems from [Marchesiello and Šnobl 2022] do not reduce to the "standard" ones.

Contents

Generalized cylindrical and spherical cases

Here we investigate integrability in two physically interesting cases from [Marchesiello and Šnobl 2022], namely the generalized cylindrical

$$X_1 = (\ell_3^A)^2 - a\ell_3^A p_3^A + c p_1^A p_3^A + \dots, \qquad X_2 = (p_3^A)^2 + \dots$$

and generalized spherical

$$X_1 = (\ell_3^A)^2 + \dots, \quad X_2 = (\ell_1^A)^2 + (\ell_2^A)^2 + (\ell_3^A)^2 + a\ell_3^A p_3^A + \dots$$

We assume that the constants a, c do not both vanish.
Integrability implies

$$\{X_1, H\} = 0, \quad \{X_2, H\} = 0, \quad \{X_1, X_2\} = 0,$$

and we solve the corresponding determining equations in the adapted coordinates.

Generalized spherical case

In the generalized spherical case only one integrable system is found,

$$\vec{B}(\vec{x}) = (0, 0, b_1)\,, \quad W(x, y) = -\frac{b_1^2}{8}(x^2 + y^2) - \frac{w_1}{x^2 + y^2}.$$

However, it is not new [Marchesiello, Šnobl, and Winternitz 2018]: It admits 2 cylindrical first order integrals

$$p_z^A, \quad \ell_z^A - \frac{b_1}{2}(x^2 + y^2),$$

therefore the additional term proportional to a in X_2 is an integral and it reduces to the standard spherical integral

$$X_2 = (\ell_1^A)^2 + (\ell_2^A)^2 + (\ell_3^A)^2 + \dots$$

Generalized cylindrical case - integrability I

Let us now focus on generalized cylindrical case. If we exclude the known standard cylindrical ($a = c = 0$) and Cartesian systems, we obtain one system for each combination of constants a, c vanishing/nonvanishing. (Some special cases were known.)

We use the shorthand $r = \sqrt{x^2 + y^2}$.

$\boldsymbol{a \neq 0, c \neq 0}$

$$B^x = b_2(ay + c), \quad B^y = -b_2 ax, \quad B^z = -(3b_2 r^2 + 2b_1),$$

$$W = b_2 \left[-\frac{b_2}{4} r^6 - \frac{b_2 a^2 + 4b_1}{8} r^4 - \frac{acb_2}{2} r^2 y + \frac{b_2 c^2}{2} x^2 + w_1 r^2 + w_2 x + w_3 y \right],$$

Generalized cylindrical case - integrability II

$a = 0, c \neq 0$

$$B^x = b_3 x + c b_2, \quad B^y = b_3 y, \quad B^z = -(2 b_3 z + 3 b_2 r^2 + b_1),$$

$$W = \frac{b_3^2}{8} z^4 - \frac{b_3 w_1}{2} z^3 - z^2 \left(\frac{b_3^2}{2} r^2 - \frac{b_2 b_3}{2} cx + \frac{w_2 b_3}{4} - \frac{w_1^2}{2} \right)$$
$$- z \left(\frac{3 b_2 b_3}{4} r^4 + \frac{b_3 b_1}{2} r^2 - \frac{b_3^2 - 2 b_2 w_1}{2} cx + \frac{w_3 b_3}{2} - \frac{w_2 w_1}{2} \right)$$
$$- \frac{b_2^2}{4} r^6 + \frac{(w_1 - b_1) b_2}{4} r^4 + \frac{b_2 b_3}{2} cx r^2 + \frac{b_2^2}{2} c^2 x^2 +$$
$$+ \left(\frac{w_1 (b_1 + w_1)}{2} - \frac{b_2 w_3}{2} \right) r^2 + \frac{b_3 (b_1 + w_1) - b_2 w_2}{2} cx,$$

Generalized cylindrical case - integrability III

$\boldsymbol{a \neq 0, c = 0}$

$$B^x = -b_2 y - b_3 \cos\left(\frac{2z}{a}\right), \quad B^y = b_2 x + b_3 \sin\left(\frac{2z}{a}\right),$$

$$B^z = \frac{b_2}{a} r^2 + b_1,$$

$$W = -(b_2 a^2 - w_1)\left[\frac{b_2}{8a^2} r^4 + \frac{2b_1 a + w_1}{8a^2} r^2 + \frac{b_3}{4}\left(x \sin\left(\frac{2z}{a}\right) + y \cos\left(\frac{2z}{a}\right)\right)\right].$$

All these fields do not contain undetermined functions, only parameters b_i, w_j, in contrast to standard cases.

Contents

Helical undulator – a superintegrable system

Due to the constrained magnetic field, it is subsequently easy to search for superintegrable systems among them. However, they are very rare, we have found only one such system so far.

$$\vec{B}(x,y,z) = \left(-b_3 \cos\left(\frac{2z}{a}\right), b_3 \sin\left(\frac{2z}{a}\right), b_1\right), \quad W(x,y,z) = 0,$$
$$\vec{A}(x,y,z) = \left(-\frac{b_3 a}{2} \cos\left(\frac{2z}{a}\right), b_1 x + \frac{b_3 a}{2} \sin\left(\frac{2z}{a}\right), 0\right).$$

It is new if both $b_i \neq 0$, otherwise the system is known from [Marchesiello, Šnobl, and Winternitz 2015]. It describes motion of an electron in a nonrelativistic limit of a helical undulator in a solenoid, neglecting the produced radiation.

Helical undulator – Integrals

It admits 3 first order integrals

$$Y_1 = p_x^A + b_1 y + \frac{b_3 a}{2} \cos\left(\frac{2z}{a}\right),$$

$$Y_2 = p_y^A - b_1 x - \frac{b_3 a}{2} \sin\left(\frac{2z}{a}\right),$$

$$Y_3 = \ell_z^A - \frac{a}{2} p_z^A - \frac{1}{2}\left[b_1 r^2 + b_3 a \sin\left(\frac{2z}{a}\right) x + b_3 a \cos\left(\frac{2z}{a}\right) y\right].$$

which do not Poisson commute

$$\{Y_1, Y_2\} = b_1, \quad \{Y_1, Y_3\} = -Y_2, \quad \{Y_2, Y_3\} = Y_1.$$

The system is integrable of the Cartesian type due to a dependent integral $X_2 = 2H - (Y_1^2 + Y_2^2 + 2b_1 Y_3) = p_z^2 + \ldots$, a Casimir invariant.

Helical undulator - nonseparable system

The helical undulator is not separable in any coordinates. It is the first such superintegrable system with a magnetic field.

This follows from [Benenti, Chanu, and Rastelli 2001]: They prove that the separation on a real Riemannian manifold must be orthogonal and is thus connected to the 11 separable systems known from the scalar case. Our system does not have the corresponding form of integrals except in the Cartesian case.

For a fixed coordinate system (e.g. Cartesian), this can be checked using the Levi-Civita separation conditions [Levi-Civita 1904] (no sum over indices $i \neq j$, $\partial_i \equiv \partial_{x^i}, \partial^j \equiv \partial_{p_j}$)

$$\partial^i \partial^j H \partial_i H \partial_j H + \partial_i \partial_j H \partial^i H \partial^j H - \partial^i \partial_j H \partial_i H \partial^j H - \partial_i \partial^j \partial^i H \partial_j H = 0,$$

which does not hold for all $i \neq j$ if $b_1 b_3 \neq 0$.

Relativistic undulator

In the absence of the electrostatic potential, $W(\vec{x}) = 0$, the relativistic Hamiltonian expressed in the instant form (see, e.g., [Heinzl and Ilderton 2017]),

$$H_{\mathrm{rel}} = \sqrt{1 + (p_j^A)^2} + W(\vec{x}),$$

is a function of the nonrelativistic Hamiltonian. Therefore the same algebra of integrals is present in the relativistic case a well.

This is crucial for the potential application, namely numerical modelling of electron bunches in helical undulators, which is a key component in free-electron lasers (FEL). (Our system describes only single electron and neglects the produced radiation.)

Implementing the undulator

Helical undulators can be implemented by redistributing the solenoidal (constant) magnetic field by a ferromagnetic helix [Balal et al. 2017] or by an array of magnets [Varfolomeev et al. 1993].

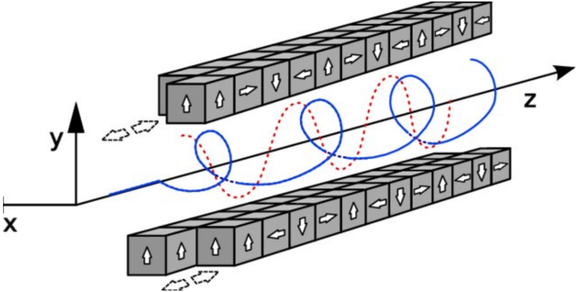

Figure: Configuration of magnets producing the linear, respectively helical, undulator field [Spezzani et al. 2011].

Conjecture – subgroup vs. nonsubgroup coordinates

We have demonstrated integrability with generalized cylindrical integrals, but excluded with generalized spherical. Generalized Cartesian were found elsewhere [Marchesiello and Šnobl 2017]

$$X_1 = p_1^2 + ap_2^2, \quad X_2 = p_2^2 + b_{ij}p_ip_j + \dots, \quad a, b_{ij} \in \mathbb{R},\ i \neq j.$$

All systems found so far separate in Cartesian or cylindrical coordinates in the limit $\vec{B} \to 0$.

We conjecture that this is because Cartesian and cylindrical coordinates are subgroup type coordinates, related to the two maximal abelian Lie subalgebras of $\mathfrak{e}(3)$, namely $\mathcal{A}_1 = \mathrm{span}\{p_1, p_2, p_3\}$ and $\mathcal{A}_2 = \mathrm{span}\{p_3, \ell_3\}$, respectively. (Spherical coordinates have $\mathcal{A} = \mathrm{span}\{\ell_3, L^2\}$.)

Contents

Conclusions

- When magnetic field is present, there can be quadratic integrals not connected to separation of variables.
- We have classified integrable systems in the generalized spherical and cylindrical cases. Only generalized cylindrical admits new systems.
- We conjecture this is connected with the presence of maximal abelian Lie subalgebras of $\mathfrak{e}(3)$ in Cartesian and cylindrical coordinates, absent in spherical coordinates.
- We have found the first (minimally) superintegrable generalized system. Its Hamilton–Jacobi equation does not separate in any coordinates systems.
- The superintegrable system models the (relativistic) helical undulator, a key component of free-electron lasers.

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Thank you for your attention!