

Eigenvalues of a perturbed periodic differential system of arbitrary order

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In $[L^2(\mathbb{R})]^m$ we consider the following eigenvalue problem:

$$(L_r + T_r)y(t) = \lambda y(t), \quad -\infty < t < \infty. \quad (1)$$

Here L_r is a self-adjoint operator corresponding to the system of ODEs

$$L_r y \equiv \sum_{\nu=1}^r p_\nu(t) y^{(\nu)} = \lambda y, \quad p_\nu(t) \in \mathbb{C}^{m \times m}, \quad p(t+1) = p(t), \quad (2)$$

and $T_r y$ is a **symmetric perturbation** of the form

$$T_r y = \sum_{k=0}^r i^k \ell_k[y], \quad \text{where} \quad (3)$$

$$\ell_{2j} = \mathcal{D}^j \eta_{2j}(t) \mathcal{D}^j, \quad \ell_{2j-1} = \mathcal{D}^{j-1} (\mathcal{D}^j \eta_{2j-1}(t) + \eta_{2j-1}^*(t) \mathcal{D}^j) \mathcal{D}^{j-1},$$

$$\mathcal{D} := \frac{d}{dt}, \quad \eta_{2j}(t), \eta_{2j-1}(t) \in \mathbb{C}^{m \times m}, \quad \eta_{2j} = \eta_{2j}^*.$$

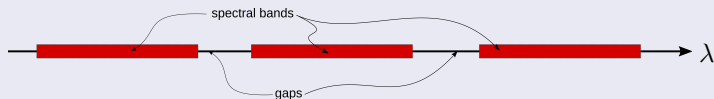
One can also write T_r in a **non-divergence form**:

$$T_r y = \sum_{\nu=0}^r \tilde{\eta}_\nu(t) y^{(\nu)}, \quad \tilde{\eta}_\nu(t) \in \mathbb{C}^{n \times n}, \quad \det(p_r + \tilde{\eta}_r) \neq 0. \quad (4)$$

The spectrum of L_r : $\sigma(L_r) = \sigma_{\text{ess}}(L_r)$

Example 1. Scalar Hill's equation

$$-y'' + p_0 y = \lambda y$$



Example 2. Vector equation

$$L_2 y = - \begin{pmatrix} y_1'' \\ y_2'' \end{pmatrix} + \begin{pmatrix} q_1(t) & 0 \\ 0 & q_2(t) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda y$$

$$\sigma(L_2) = \sigma(\ell_1) \cup \sigma(\ell_2), \text{ where } \ell_k y_k = -y_k'' + q_k(t)y_k = \lambda y_k, \quad k = 1, 2$$



Remark

In the scalar case one has $\sigma(L_1) = (-\infty, \infty)$.

The operator L_r is **not lower semibounded** either if r is odd or if r is even and $p_r(t)$ is an indefinite matrix-function.

Theorem 1 [K. (1973, 1977)]

1. (T_r is in divergence form (3))

Let L_r be lower-semibounded (hence, $r = 2n$ and $(-1)^n p_{2n}(t) > 0$), while the coefficients of T_{2n} satisfy

$$\begin{aligned} \exists \alpha, \beta \in \mathbb{R} : \quad 0 < \alpha I_m < (-1)^n p_{2n}(t) + \eta_{2n}(t) < \beta I_m \\ \exists \gamma \in \mathbb{R} : \quad |\eta_{2n-1}(t)| < \gamma, \\ \lim_{|x| \rightarrow \infty} \int_x^{x+1} |\eta_j(t)| dt = 0, \quad j = 0, \dots, 2n. \end{aligned}$$

Then $\sigma_{\text{ess}}(L_{2n} + T_{2n}) = \sigma_{\text{ess}}(L_{2n})$.

2. (T_r is in non-divergence form (4))

Let

$$\lim_{|t| \rightarrow \infty} |\tilde{\eta}_r(t)| = 0, \quad (5)$$

$$\lim_{|x| \rightarrow \infty} \int_x^{x+1} |\tilde{\eta}_\nu(t)|^2 dt = 0, \quad \nu = 0, \dots, r-1. \quad (6)$$

Then $\sigma_{\text{ess}}(L_r + T_r) = \sigma_{\text{ess}}(L_r)$.

Discrete spectrum in gaps of essential spectrum

- $\sigma(L_r) = \{\lambda : \text{Eqn. } L_r y = \lambda y \text{ has } \textbf{multipliers} \text{ with absolute value } 1\}$
- **Multiplicators** are eigenvalues of the Wronski matrix $\|Y_i^{(k-1)}(1, \lambda)\|_{i,k=1}^r$ for Eqn. $L_r y = \lambda y$ at $t = 1$; here $Y_t(t, \lambda)$ are $m \times m$ matrix solutions of Eqn. $L_r y = \lambda y$ such that $\|Y_i^{(k-1)}(0, \lambda)\| = I_{mr}$

Theorem 2 [K. (1977)]

Let μ_0 be one of the endpoints of a spectral gap Δ of the operator L_r . Then the eigenvalues of the problem (1) lying in Δ **do not accumulate** at the point μ_0 if (5) holds and

$$\overline{\lim}_{|x| \rightarrow \infty} \left| x^{2p-1} \sum_{\nu=0}^r \int_x^{\infty \operatorname{sgn}(x)} (|\tilde{\eta}_\nu(t)| + |\tilde{\eta}_\nu(t)|^2) dt \right| < \delta,$$

where $2p$ is the size of the largest Jordan block, which correspond to unimodular multipliers of Eqn. (2) for $\lambda = \mu_0$, and the constant $\delta = \delta(L_r, \mu_0)$.

Remark

It is known that the sizes of Jordan blocks corresponding to unimodular multipliers of Eqn. (2) at the endpoints of spectral gaps are **even**.

Main results: conditions leading to **infinitely many** eigenvalues in a gap

In the theorem below we assume that $\eta_k(t) \equiv 0$ for $k > 2n$, where $n = [\frac{r}{2}]$.

Theorem 3 [K. (2022)]

Let $\Delta = (\lambda_-, \lambda_+)$ be either finite or infinite gap in $\sigma_{\text{ess}}(L_r)$.

Then $\sigma_{\text{ess}}(L_r + T_r) = \sigma_{\text{ess}}(L_r)$, and, if $|\lambda_{\pm}| < \infty$, the eigenvalues of $L_r + T_r$ lying in Δ **accumulate** to λ_{\pm} , provided the following two conditions hold:

1° $\lim_{|x| \rightarrow \infty} \int_x^{x+1} (|\eta_{2k}^{(i)}(t)| + |\eta_{2k-1}^{(i)}(t)|^2) dt = 0, \quad i = 0, 1, \dots, k + r.$

2° There is such t_0 that for any $t \leq t_0 < 0$ or $t \geq t_0 > 0$ and for any $y \in C_0^\infty(\mathbb{R})$ with $\text{supp}(y) \subset (-\infty, t_0)$ or $\text{supp}(y) \subset (t_0, \infty)$ one has

$$\exists k_{\pm} \geq 0 : \quad \pm \int_{-\infty}^{\infty} (T_r y, y) dt \leq \pm \int_{-\infty}^{\infty} (\tau_{\pm} y, y) dt, \quad (7)$$

where $\tau_{\pm} = \mp C_{\pm}(-1)^{k_{\pm}} \mathcal{D}^{k_{\pm}} \frac{1}{t^{2p_{\pm}}} 1_m \mathcal{D}^{k_{\pm}}$ with $2p_{\pm}$ being the maximal of the orders of Jordan blocks associated with unimodular multipliers of equation

$$L_r y = \lambda_{\pm} y, \quad (8)$$

at that, if $k_{\pm} > 0$, only such multipliers are taken into consideration that among the corresponding to them Floquet solutions there exists non-constant ones;

$C_{\pm} = C_{\pm}(L_r, \lambda_{\pm}, k_{\pm}) > 0$ are some constants.

We build $m(n+1) \times m(n+1)$ matrix from the coefficients of T_r :

$$H(t) = \begin{pmatrix} \eta_0 & i\eta_1^* & 0 & \dots & \dots & \dots \\ -i\eta_1 & \eta_2 & i\eta_3^* & \dots & \dots & \dots \\ 0 & -i\eta_3 & \eta_4 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \eta_{2(n-1)} & \eta_{2n-1}^* \\ \dots & \dots & \dots & \dots & -i\eta_{2n-1} & \eta_{2n} \end{pmatrix}$$

Remark

Condition (7) holds, in particular, if for $t \leq t_0 < 0$ or for $t \geq t_0 > 0$ one has

$$\pm \left(\text{diag}(0_m, \dots, 0_m, \frac{\mp C_{\pm}}{t^{2p_{\pm}}} 1_m, 0, \dots, 0_m) - H(t) \right) \geq 0. \quad (9)$$

This inequality holds for λ_+ if, for example,

$$\nu(H(t)) \leq -\frac{C_+}{t^{2p_+}},$$

where $\nu(H(t))$ is the largest eigenvalue of the matrix $H(t)$; similar statement holds for λ_- .

Remark

The assumption on unimodular multipliers standing at the definition of p_{\pm} (that, if $k_{\pm} > 0$, there exist non-constant Floquet solutions corresponding to these multipliers) cannot be omitted, as the example below demonstrates.

Example. We consider the scalar operator $L_4 = y^{(4)}$. One has $\sigma(L_4) = [0, \infty)$, thus $\lambda_+ = 0$. All multipliers of the equation $y^{(4)} = 0$ equal to 1; the corresponding Jordan blocks are of the size 4; thus $p_+ = 4$. The perturbed operator

$$(L_4 + T_2)y = y^{(4)} + \underbrace{\left[- \left(\frac{C}{t^4} y' \right)' \right]}_{T_2} \quad (|t| > 1)$$

does not have infinite many eigenvalues below 0 for any C , although the inequality (9) holds with $p_+ = 2$, $k_+ = 1 > 0$, $-C_+ \geq C$. The equation $y^{(4)} = 0$ has only one Floquet solution $y \equiv \text{const}$.

Remark

One can show that there are no constant Floquet solutions at the endpoints of gaps located sufficiently far away from zero.

Discrete spectrum in a semi-infinite gap

In the case of semi-infinite gap Theorem 3 can be specified. Namely, we can “distinguish” a part in the periodic operator L_r , which corresponds to unimodular multipliers, and **generate infinitely many eigenvalues in this gap perturbing only this part**, and not the whole operator.

Let $\inf L_r = \lambda_0 > -\infty$ (consequently, $r = 2n$ and the leading coefficient p_{2n} satisfies $(-1)^n p_{2n}(t) > 0$).

To formulate the next Theorem 4 we first present some auxiliary results.

Proposition

Eqn. (1) for $r = 2n$ reduces to the system

$$x' = \begin{pmatrix} h_1(t) & h_2(t) \\ h_3(t) & -h_1^*(t) \end{pmatrix} x, \quad h_1, h_2 = h_2^*, h_3 = h_3^* \in \mathbb{C}^{mn \times mn} \quad (10)$$

by means of the substitution

$$x = \text{col}\{y, y', \dots, y^{(n-1)}, y^{[2n-1]}, \dots, y^{[n]}\},$$

where $y^{[j]}$ are quasi-derivatives

Theorem [K.]

Let λ_0 be the edge of a semi-infinite gap. Then the system (11), corresponding to Eqn. (2) at $\lambda = \lambda_0$, possesses $2mn \times mn$ -matrix Floquet-type solution:

$$\begin{pmatrix} U(t) \\ V(t) \end{pmatrix}, \quad U, V \in \mathbb{C}^{mn \times mn},$$

where $U(t) = Z(t) \exp(\Lambda t)$, $Z(t+1) = Z(t)$, $\det Z(t) \neq 0$.

For the matrix Λ the sizes of Jordan blocks corresponding to imaginary characteristic numbers are equal to the half of the sizes of the Jordan blocks corresponding to unimodular multipliers of Eqn. (2) for $\lambda = \lambda_0$.

Let the matrix S reduces Λ to the following block-diagonal form:

$$S^{-1} \Lambda S = \text{diag}(\Lambda_1, \Lambda_2), \quad \Lambda_1 \in \mathbb{C}^{q \times q}, \Lambda_2 \in \mathbb{C}^{\ell \times \ell},$$

where $\sigma(\Lambda_1) \subset i\mathbb{R}$, $\sigma(\Lambda_1) \cap i\mathbb{R} = \emptyset$.

Let $\Lambda_1 = D + N$ be the Dunford decomposition of the matrix Λ_1 , where D is a diagonalizable matrix and N is a nilpotent matrix.

Denote

$$\tilde{\Lambda} = \text{diag}(N, \Lambda_2), \quad \tilde{Z} = Z(t)S \cdot \exp(\text{diag}(D, I_\ell)t).$$

Let e_j^i ($i = 1, \dots, q_j$) be the system of vectors consisting of

- eigenvectors e_j^1 of the matrix $\tilde{\Lambda}$, which correspond to the eigenvalue 0,
- and generalized eigenvectors of $\tilde{\Lambda}$, i.e., $\tilde{\Lambda}e_j^i = e_j^{i-1}$, $i = 1, \dots, q_j$, $e_j^0 = 0$.

Then we define the subspaces $\mathfrak{M}^s(e_{j_1}^1, \dots, e_{j_k}^1)$, $1 \leq s \leq \min\{q_{j_1}, \dots, q_{j_k}\}$:

- $\mathfrak{M}^1(e_{j_1}^1, \dots, e_{j_k}^1) := \text{span}\{e_{j_1}^1, \dots, e_{j_k}^1\}$,
- $\mathfrak{M}^s(e_{j_1}^1, \dots, e_{j_k}^1) := \mathfrak{M}^{s-1}(e_{j_1}^1, \dots, e_{j_k}^1) + \text{span}\{e_{j_1}^s, \dots, e_{j_k}^s\}$

With the perturbation $T_{2(n-1)}$ we associate the following block diagonal matrix consisting of its coefficients:

$$H_1(t) = \begin{pmatrix} \eta_0 & i\eta_1^* & 0 & \dots & \dots & \dots \\ -i\eta_1 & \eta_2 & i\eta_3^* & \dots & \dots & \dots \\ 0 & -i\eta_3 & \eta_4 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \eta_{2(n-2)} & \eta_{2n-1}^* \\ \dots & \dots & \dots & \dots & -i\eta_{2n-1} & \eta_{2(n-1)} \end{pmatrix}$$

and $q \times q$ matrix $\tilde{H}_{11}(t)$ such that

$$\tilde{Z}^*(t)H_1(t)\tilde{Z}(t) = \begin{pmatrix} \tilde{H}_{11}(t) & * \\ * & * \end{pmatrix} \quad (11)$$

Finally, we assume that for $T_{2(n-1)}$ the condition 1° from Theorem 1 fulfills.

Theorem 4 [K. (2022)]

The eigenvalues of the operator $L_{2n} + T_{2(n-1)}$, which belong to $(-\infty, \lambda)$, **do accumulate at** λ if there is such t_0 that for any $t \leq t_0 < 0$ or $t \geq t_0 > 0$ there exists two subspaces $\mathfrak{M}^1(e_{j_1}^1, \dots, e_{j_k}^1)$, $\mathfrak{M}^s(e_{j_1}^1, \dots, e_{j_k}^1)$ such that

$$1^\circ \quad (\tilde{H}_{11}\xi, \xi) \leq 0, \quad \forall \xi \in \mathfrak{M}^s(e_{j_1}^1, \dots, e_{j_k}^1).$$

$$2^\circ \quad \min_{\xi: \sum_{\beta=1}^k |\xi_\beta|^2 = 1} \overline{\lim}_{x \rightarrow (\operatorname{sgn} t_0)\infty} \left| x^{2s-1} \int_x^{(\operatorname{sgn} t_0)\infty} (\tilde{H}_{11}(t)\xi, \xi) dt \right| > K,$$

$$\text{where } \xi = \sum_{\beta=1}^k \xi_\beta e_{j_\beta}^1, \quad K = K(L_{2n}, \lambda_0, \tilde{Z}, \mathfrak{M}^s) > 0.$$

Corollary

The eigenvalues of the operator $L_{2n} + T_{2(n-1)}$, which belong to $(-\infty, \lambda_0)$, **do accumulate** at λ_0 if there is such t_0 that for any $t \leq t_0 < 0$ or $t \geq t_0 > 0$:

$$\tilde{H}_{11}(t) \leq 0 \quad \text{and} \quad \overline{\lim}_{x \rightarrow (\operatorname{sgn} t_0)\infty} \left| x^{2p-1} \int_x^{(\operatorname{sgn} t_0)\infty} \mu(-\tilde{H}_{11}(t)) dt \right| = \infty, \quad (12)$$

where $\mu(\cdot)$ is the smallest eigenvalue, $2p$ is the size of the largest Jordan block, which correspond to unimodular multipliers of Eqn. (2) for $\lambda = \lambda_0$.

One gets more rough result if in (12) $\tilde{H}_{11}(t)$ is replaced by $H_1(t)$.

Thank you for your attention!