A discrete quantum wave function for the inverted oscillator

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Abstract

While physical properties of the simple harmonic oscillator are well known, the study for the inverted oscillator is also necessary because it can be applied to many physical systems. The characteristics of the inverted oscillator is quite different from that of the simple harmonic oscillator. The wave packet in the inverted oscillator associated with the usual plane wave solutions is unbound, their eigenstates are not square-integrable.
A general quantum mechanical properties of a mechanical system can be studied in terms of the invariant operators that are developed by Lewis and Riesenfeld. The invariant operator is useful for evaluating the eigenvalue problem of mechanical systems.
The wave solutions that have discrete eigen spectrum are interesting. We focus in this talk on the quantum solutions with discrete eigen spectrum. The characteristics relevant to discrete eigen spectrum will be analyzed in detail. The difference between discrete eigen spectrum and the continuous one will be compared.

I) Introduction

The physical properties of the simple harmonic oscillator are well known, the study for the inverted oscillator is also necessary because it can be applied to many physical systems. Some of the application of the inverted oscillator are 2D string theory associated with non-interacting fermions, fundamental inflation models in cosmology fission dynamics, nonlinear optical phenomena, string theory, etc. The characteristics of the inverted oscillator is quite different from that of the simple harmonic oscillator. The wave packet in the inverted oscillator associated with the usual plane wave solutions is unbound, their eigenstates are not square-integrable.

A general quantum mechanical properties of a mechanical system can be studied in terms of the invariant operators that are developed by Lewis and Riesenfeld. The invariant operator is useful for evaluating the eigenvalue problem of mechanical systems.

The eigen solutions of the invariant operator are the same as the wave function when we do not mind the phase factors. The phase factors can also be obtained by the aid of the Schrödinger equation. This is the reason why the invariant operator method is powerful in the quantum treatment of mechanical systems.

The wave solutions that have discrete eigen spectrum are interesting. We focus in this work the quantum solutions with discrete eigen spectrum. The characteristics relevant to discrete eigen spectrum will be analyzed in detail. The difference between discrete eigen spectrum and the continuous one will be compared.

II) Invariant operator

The inverted oscillator is described by the Hamiltonian

$$H = \frac{1}{2}\left(\frac{p^2}{m} - m\omega^2 q^2\right). \tag{1}$$

where m is mass and ω is the repulsion parameter. Therefore, to solve the Schrödinger equation associated with the Hamiltonian (1)

$$i\hbar\frac{\partial}{\partial t}\psi(q,t) = H\psi(q,t), \tag{2}$$

Introducing the general hermitian invariant operator $I(t)$

$$I(t) = \frac{1}{2}\left[\alpha(t)p^2 + \beta(t)q^2 + \gamma(t)\left\{q,p\right\}\right], \tag{3}$$

where $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are real time dependents coefficients which will be determined later.
The invariant operator $I(t)$ should satisfy the Liouville-Von Neumann equation

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + \frac{1}{i\hbar}[I, H] = 0. \tag{4}$$

By inserting the Hamiltonian and the invariant operators defined respectively by equations (1) and (3) in the Liouville-Von Neumann equation (4), we obtain a system of linear first order differential equations for the coefficients $\alpha(t)$, $\beta(t)$ and $\gamma(t)$

$$\begin{aligned} \frac{1}{2}\left[\dot{\alpha}p^2 + \dot{\beta}q^2 + \dot{\gamma}\{q,p\}\right] &= -\frac{\beta}{2m}\{q,p\} - \frac{\gamma}{m}p^2 - \frac{\eta}{m}p \\ &-\frac{1}{2}\alpha m\omega^2\{q,p\} - \gamma m\omega^2 q^2 - \delta m\omega^2 q \end{aligned} \tag{5}$$

So, the equation (5) give us

$$\begin{cases} \beta - \beta_0 = m^2\omega^2\left(\alpha - \alpha_0\right) \\ \ddot{\gamma} - 4\omega^2\gamma = 0 \\ \ddot{\alpha} - 4\omega^2\alpha = \frac{2}{m^2}\left[\beta_0 - m^2\omega^2\alpha_0\right] \\ \ddot{\beta} - 4\omega^2\beta = 2\omega^2\left[m^2\omega^2\alpha_0 - \beta_0\right] \end{cases} \tag{6}$$

and

$$\alpha\beta - \gamma^2 = \omega_0^2 = cst \tag{7}$$

III) The Schrödinger equation solutions

From Eq. (2)one can see that if $\psi_\lambda(q, t)$ is a solution of the time-dependent Schrödinger equation, any function defined by $\varphi_\lambda = (I\psi_\lambda)$ will also be. In particular, one can choose $\psi_\lambda(q, t)$ as being the eigenfunction of $I(t)$. Therefore, this suggests that the solution of the time dependent Schrodinger equation has the form

$$\psi_\lambda = \exp\left(\frac{i}{\hbar}\delta_\lambda\right)\varphi_\lambda \tag{8}$$

where the phases $\delta_\lambda(t)$ are given by

$$\hbar\delta_\lambda(t) = \langle\varphi_\lambda(q,t)| \left(i\hbar\frac{\partial}{\partial t} - H\right) |\varphi_\lambda(q,t)\rangle \,. \tag{9}$$

Let's look at the eigenvalue equation to determine the eigenfunctions $\varphi_\lambda\,(q,t)$ and the eigenvalues λ

$$I(t)\varphi_\lambda\,(q,t) = \lambda\varphi_\lambda\,(q,t)\,. \tag{10}$$

To simplify this eigenvalues equation, we introduce the following unitary transformation

$$\varphi_\lambda\,(q,t) = U\varphi'_\lambda\,(q,t)\,, \tag{11}$$

where the unitary operator $U = U_1 U_2$ is given by

$$\begin{aligned} U_1 &= \exp\left[-\frac{i\gamma}{2\hbar\alpha}q^2\right], \\ U_2 &= \exp\left[-\frac{i}{4\hbar}\ln\left(\alpha\right)\left(qp+pq\right)\right], \end{aligned} \tag{12}$$

having the following properties

$$
\begin{aligned}
U_1^{+} q U_1 &= q, \qquad U_1^{+} p U_1 = p - \frac{\gamma}{\alpha} q \\
U_2^{+} q U_2 &= \sqrt{\alpha} q \ , \qquad U_2^{+} p U_2 = \sqrt{\frac{1}{\alpha}} p
\end{aligned}
\tag{13}
$$

Under this unitary transformation, the transformed invariant $I'(t)$ is simplied into

$$
I' = U_2^{+} U_1^{+} I(t) U_1 U_2 = \frac{1}{2} p^2 + \frac{1}{2} \omega_0^2 q^2 \tag{14}
$$

The fact that ω_0^2 is constant enables us to investigate the system for three separate cases, i.e., the cases that $\omega_0^2 < 0$, $\omega_0^2 = 0$, and $\omega_0^2 > 0$.
At first, the case of $\omega_0^2 < 0$ is not so interesting because the transformed system is the same as the original system with unit mass. The spectrum of wave functions for this case corresponds to continuous ones. In the case $\omega_0^2 = 0$, the transformed system corresponds to a free particle where the spectrum of wave functions is also continuous. In fact, the case we emphasis in this research is $\omega_0^2 > 0$. The transformed system in this case is well known harmonic oscillator that has discrete energy spectrum.

Case 1: $\omega_0 > 0$ discrete spectrum

It is a stationary Schrödinger equation of a harmonic oscillator of unit mass and frequency ω_0. The solution is well known

$$\varphi'_{\lambda_n}(q,t) = \left[\frac{1}{n!2^n\sqrt{\pi\hbar}}\right]^{\frac{1}{2}} \exp\left[-\frac{\omega_0 q^2}{2\hbar}\right] H_n\left[\sqrt{\frac{\omega_0}{\hbar}}q\right], \tag{15}$$

with eigenvalues

$$\lambda_n = \hbar\omega_0\left(n+\frac{1}{2}\right), \tag{16}$$

and H_n are the n order Hermite polynomials. By grouping the intermediate above results, we obtain the eigenfunctions of $I(t)$

$$\varphi_{\lambda_n}(q,t) = \left[\frac{1}{n!2^n\sqrt{\pi\hbar\alpha}}\right]^{\frac{1}{2}} \exp\left[-\frac{\gamma}{2\hbar\alpha}q^2\right] H_n\left[\sqrt{\frac{\omega_0}{\hbar\alpha}}(q)\right]. \tag{17}$$

By using the fact that the phases $\delta_n(t)$ are given by

$$\hbar\dot{\delta}_{\lambda_n}(t) = \left\langle \varphi_{\lambda_n}(q,t)\right| \left(i\hbar\frac{\partial}{\partial t} - H\right) \left|\varphi_{\lambda_n}(q,t)\right\rangle, \tag{18}$$

having $\varphi_{\lambda_n}(q,t) = U\varphi'_{\lambda_n}(q,t)$ and eliminating ω^2 from the Hamiltonian one obtains the phases

$$\delta_{\lambda_n}(t) = -\frac{\omega_0}{m}\left(n+\frac{1}{2}\right)\int_0^t \frac{d\tau}{\alpha(\tau)}. \tag{19}$$

Hence the solution of the Schrödinger equation

$$\begin{aligned}\psi_{\lambda_n}(q,t) &= \left[\frac{1}{n!2^n\sqrt{\pi\hbar\alpha}}\right]^{\frac{1}{2}} \exp\left[-\frac{i\omega_0}{m}\left(n+\frac{1}{2}\right)\int_0^t \frac{d\tau}{\alpha(\tau)}\right] \\ &\quad \times \exp\left[-\frac{\gamma}{2\hbar\alpha}q^2\right] H_n\left[\sqrt{\frac{\omega_0}{\hbar\alpha}}q)\right]\end{aligned} \tag{20}$$

therefore, the general solution is

$$\Psi(q,t) = \sum_n C_n\psi_{\lambda_n}(q,t). \tag{21}$$

Case 2: $\omega_0 = \mathbf{0}$ Free Particle: the transformed system corresponds to the case of a free particle. The spectrum of wave functions for this case is continuous.

The eigenvalue equation $\left(I' = \frac{1}{2}p^2\right)$ is a stationary Schrödinger equation of unit mass free particle. The solution of this equation is a plane wave. In this case's

1) **Airy function**

We can build the Airy solutions starting from the plane wave

$$\varphi'_\lambda(q,t) = \mathrm{Ai}\left[\frac{B}{\hbar^{\frac{2}{3}}}\left(q - \frac{B^3 t^2}{4}\right)\right]\exp\left[\frac{iB^3 t}{2\hbar}\left(q - \frac{B^3 t^2}{6}\right)\right],$$

with eigenvalues

$$\lambda = \frac{\hbar^2 k^2}{2}. \tag{22}$$

By grouping the intermediate above results, we obtain the eigenfunctions of $I(t)$

$$\varphi_\lambda(q,t) = U_1 U_2 \varphi'_\lambda(q,t)$$

2) **Gaussian wave packet**

One can also build Gaussian solutions starting from the plane wave

$$\begin{aligned}\varphi'_\lambda(q,t) &= \frac{1}{\sqrt{2\pi}}\left(\frac{a^2}{2\pi}\right)^{\frac{1}{4}} \exp\left[\frac{i}{\hbar}k_0\left(q-\frac{k_0 t}{2}\right)\right] \\ &\times \exp\left[-\frac{1}{4\alpha\hbar^2}(q-k_0 t)^2\right] \\ &\times \int_{-\infty}^{+\infty} \exp\left\{-\alpha\left[k-\frac{i}{2\alpha\hbar}(x-k_0 t)\right]^2\right\} dk, \quad (23)\end{aligned}$$

where $\alpha = \left[\frac{a^2}{4}+\frac{it}{2\hbar}\right]$. After integration we have

$$\varphi'_\lambda(q,t) = \frac{1}{\sqrt{\alpha}}\left(\frac{a^2}{8\pi}\right)^{\frac{1}{4}} \exp\left[\frac{i}{\hbar}k_0\left(q-\frac{k_0 t}{2}\right)\right] \exp\left[-\frac{1}{4\alpha\hbar^2}(q-k_0 t)^2\right]. \quad (24)$$

By grouping the intermediate above results, the eigenfunctions of $I(t)$ are $\varphi_\lambda(q,t) = U_1 U_2 \varphi'_\lambda(q,t)$.

Case 3: $\omega_0 < \mathbf{0}$ inverted oscillator: this is reduced to the started Hamiltonian and it is not interesting.

Thank You For Your Attention