Solvable potentials from Heun type euations and their symmetries

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AAMP'19, 6–9 September 2022, Prague

1. Exactly solvable potentials in general

Why are they important... especially if they are complex?

How are they generated?

Variable transformation $z(x)$: Schrödinger eq. $\Longrightarrow$ diff. eq. of special function $F(z)$

SUSYQM: Known solvable potential a $\Longrightarrow$ new solvable potentials

How are they classified?

Using the special function $F(z)$, the variable transformation $z(x)$ and SUSYQM

2. Natanzon potentials: the (confluent) hypergeometric differential equation

Adaptating the techniques ${}_2F_1(a, b; c; z)$ and ${}_1F_1(a; c; z)$

Bound states Jacobi and generalized Laguerre polynomials

Shape invariance

3. Beyond the Natanzon class: the Heun type differential equations

The sextic oscillator: introduced as QES potential

The rationally extended harmonic oscillator: SUSY partner of the HO

1. Exactly solvable potentials in general

Milestones of generating solvable potentials

1940:	Factorization method	*Schrödinger*
1951:	Systematic application of the factorization method	*Infeld and Hull*
1962:	A variable transformation method	*Bhattacharjie and Sudarshan*
1971:	Systematic application of the transformation method	*Natanzon*
1981:	SUSYQM: a reformulation of the factorization method	*Witten*

Note the periodicity of $\simeq$ 11 years

Other examples for phenomena with $\simeq$ 11 year periodic activity maxima

The Sun

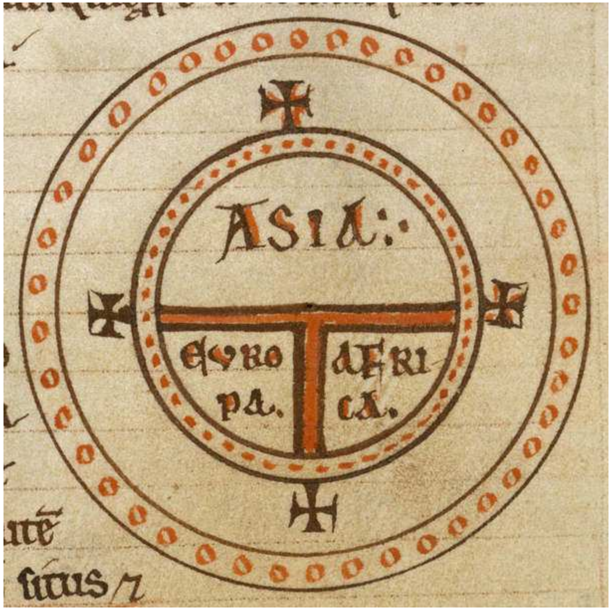
Other examples for phenomena with $\simeq$ 11 year periodic activity maxima

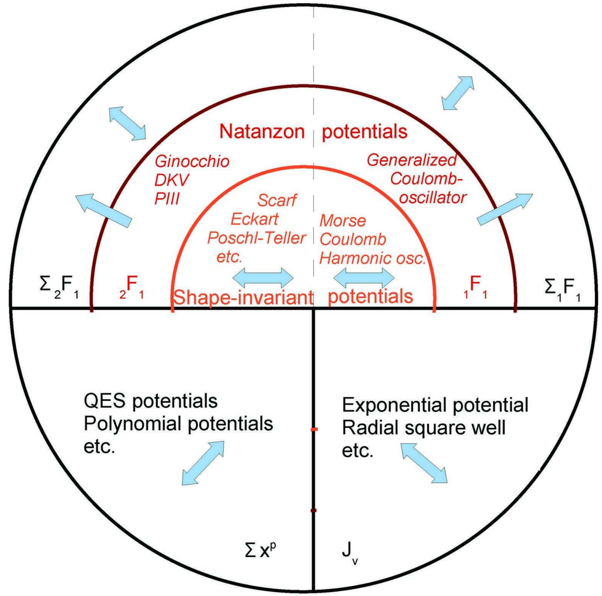
The Sun

The Soviet/Russian army

1945	Berlin
1956	Budapest
1968	Prague
1979	Kabul
1991	Moscow
2000	Grozny
2014	Crimea
2022	...

The world map of solvable potentials





Any more continents or islands?

The main territories in the map

${}_2F_1$: Natanzon class, solved by ${}_2F_1$ in general, by $P_n^{(\alpha,\beta)}(z)$ for bound states

${}_1F_1$: Natanzon confluent class, solved by ${}_1F_1$ in general, by $L_n^{(\alpha)}(z)$ for bound states

Shape-invariant: Natanzon (confluent) subclass, closed under a SUSY transformation

Σ_2F_1 and Σ_1F_1: solutions in terms of the linear combination of several (confluent) hypergeometric functions

Non-SI SUSY partners of Natanzon (confluent) potentials

Potentials solved by exceptional orthogonal polynomials, a new type of SI

Solutions containing both independent solutions gen. Woods–Saxon

J_ν: potentials solved by Bessel functions

Σx^p: Quasi-exactly solvable potentials: exact solutions up to a finite n

The variable transformation method

Bhattacharjie and Sudarshan 1962

Schrödinger eq. $\Longrightarrow$ differential equation of special function F

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + (E - V(x))\psi(x) = 0 \qquad \text{insert} \qquad \psi(x) = \mathbf{f}(x)F(\mathbf{z}(x))$$

and compare with

$$\frac{\mathrm{d}^2 F}{\mathrm{d}\mathbf{z}^2} + Q(\mathbf{z})\frac{\mathrm{d}F}{\mathrm{d}\mathbf{z}} + R(\mathbf{z})F(\mathbf{z}) = 0$$

to get

$$E - V(x) = \frac{\mathbf{z}'''(x)}{2\mathbf{z}'(x)} - \frac{3}{4}\left(\frac{\mathbf{z}''(x)}{\mathbf{z}'(x)}\right)^2 + (\mathbf{z}'(x))^2\left(R(\mathbf{z}(x)) - \frac{1}{2}\frac{\mathrm{d}Q(\mathbf{z})}{\mathrm{d}\mathbf{z}} - \frac{1}{4}Q^2(\mathbf{z}(x))\right) .$$

Schwartzian derivatve terms $\qquad$ E and the main potential terms

Connection to SUSYQM:

$$W(x) = -\frac{1}{2}Q(z(x))z'(x) + \frac{z''(x)}{2z'(x)}$$

The solutions are

$$\psi(x) \sim (\mathbf{z}'(x))^{-\frac{1}{2}} \exp\left(\frac{1}{2}\int^{\mathbf{z}(x)} Q(\mathbf{z})\mathrm{d}\mathbf{z}\right) F(\mathbf{z}(x)) \ .$$

The yet unknown $\mathbf{z}(x)$ can be obtained from a differential equation

$$\boxed{\left(\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}x}\right)^2 \phi(\mathbf{z}) = C \ ,}$$

by direct integration

$$\int \phi^{1/2}(\mathbf{z})\mathrm{d}\mathbf{z} = C^{1/2}x + \gamma \ .$$

γ: integration constant, coordinate shift

This is to generate a **constant term** on the r.h.s. of $E - V(x) = ...$

The special function F and the variable transformation $\mathbf{z}(x)$ **determines everything**

Note: sometimes only $x(\mathbf{z})$ can be determined $\Longrightarrow$ implicit potentials

2. Natanzon potentials: the (confluent) hypergeometric differential equation

Natanzon-class potentials *Natanzon 1971*

Employ the method to the hypergeometric function ${}_2F_1(a,b;c;z)$

Or the confluent hypergeometric function ${}_1F_1(a,c;z)$

For bound states these functions reduce to orthogonal polynomials:

$$ {}_2F_1(a,b;c;z) \Longrightarrow P_n^{(\alpha,\beta)}(1-2z) \qquad \text{for} \quad a=-n \quad \text{or} \quad b=-n $$

Jacobi polynomial

$$ {}_1F_1(a,c;z) \Longrightarrow L_n^{(\alpha)}(z) \qquad \text{for} \quad a=-n $$

Generalized Laguerre polynomial

In what follows we employ Jacobi polynomials

They can be adapted better to $\mathcal{PT}$-symmetric QM:

Their argument exhibits $\mathcal{PT}$ symmetry

Apply the method to the Jacobi polynomials: $F(z) = P_n^{(\alpha,\beta)}(z)$

$$
\begin{aligned}
E - V(x) \quad = \quad & \frac{z'''(x)}{2z'(x)} - \frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2 + \frac{(z'(x))^2}{1-z^2(x)}\left(n+\frac{\alpha+\beta}{2}\right)\left(n+\frac{\alpha+\beta}{2}+1\right) \\
& + \frac{(z'(x))^2}{(1-z^2(x))^2}\left[1-\left(\frac{\alpha+\beta}{2}\right)^2-\left(\frac{\alpha-\beta}{2}\right)^2\right] \\
& - \frac{2z(x)(z'(x))^2}{(1-z^2(x))^2}\left(\frac{\alpha+\beta}{2}\right)\left(\frac{\alpha-\beta}{2}\right) .
\end{aligned}
$$

The solutions are

$$
\psi(x) \sim (z'(x))^{-\frac{1}{2}}(1+z(x))^{\frac{\beta+1}{2}}(1-z(x))^{\frac{\alpha+1}{2}} P_n^{(\alpha,\beta)}(z(x)) \ .
$$

The yet unknown $\mathbf{z}(x)$ can be obtained from a differential equation

$$
\boxed{\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 \phi(z) \equiv \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 \frac{p_{\mathrm{I}}(1-z^2)+p_{\mathrm{II}}+p_{\mathrm{III}}z}{(1-z^2)^2} = C \ .}
$$

by direct integration

$$
\int \phi^{1/2}(\mathbf{z})\mathrm{d}\mathbf{z} = C^{1/2}x + \epsilon \ .
$$

ϵ: integration constant, coordinate shift

Separating the constant E term, the potential is

$$V(x) = -\frac{z'''(x)}{2z'(x)} + \frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2 + \frac{C}{\phi(z)}\left[s_I(1-z^2(x)) + s_{II} + s_{III}z(x)\right] .$$

The solutions are

$$\psi(x) \sim [\phi(z(x))]^{\frac{1}{4}}(1+z(x))^{\frac{\beta}{2}}(1-z(x))^{\frac{\alpha}{2}}P_n^{(\alpha,\beta)}(z(x)) .$$

$$(n+\tfrac{1}{2}+\omega)^2 - \tfrac{1}{4} + s_I - p_I\frac{E_n}{C} = 0$$

$$(1-\omega^2-\rho^2) + s_{II} - p_{II}\frac{E_n}{C} = 0$$

$$-2\omega\rho + s_{III} - p_{III}\frac{E_n}{C} = 0$$

$$\omega = (\alpha+\beta)/2 \text{ and } \rho = (\alpha-\beta)/2$$

Solving the problem: chose $p_i \Longrightarrow$ get $z(x) \Longrightarrow$ express $E_n \Longrightarrow$ get $V(x)$

About the origin of the parameters:

s_i: from the parameters of the polynomial α and β

p_i: (including also C and ϵ) from the variable transformation function $z(x)$

What about the role of SUSYQM?

New potentials are generated from old ones

The solutions of the new potentials are obtained from those of the old ones:

$$\psi_+(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x} + W(x) \right) \psi_-(x)$$

If $\psi_-(x)$ contains $P_n^{(\alpha,\beta)}(z(x))$ then ...

...$\psi_+(x)$ contains $P_n^{(\alpha,\beta)}(z(x))$ AND $P_n^{(\alpha,\beta)}(z(x))'$

...so it is a linear combination of several Jacobi polynomials

BUT sometimes recursion relations help to restore the original structure

Shape-invariant potentials: they correspond to simple choice of $z(x)$, i.e. p_i

The list of (real) shape-invariant potentials ($a = 1$, $C = \pm 1$)

$(z')^2 =$ (Class)	$V(x)$	$x \in$	Name
$C(1-z^2)$ (PI)	$(B^2 - A^2 - A)\text{sech}^2(x) + B(2A+1)\text{sech}(x)\tanh(x)$	$(-\infty, \infty)$	Scarf II
	$(B^2 + A^2 + A)\text{cosech}^2(x) - B(2A+1)\text{cosech}(x)\coth(x)$	$[0, \infty)$	gen. Pöschl–Teller
	$(B^2 + A^2 - A)\text{cosec}^2(ax) - B(2A-1)\text{cosec}(x)\cot(x)$	$[0, \pi]$	Scarf I
	$A(A-1)\sec^2(x) + B(B-1)\text{cosec}^2(x)$	$[0, \pi/2]$	Pöschl–Teller I
	$-A(A+1)\text{sech}^2(x) + B(B-1)\text{cosech}^2(x)$	$[0, \infty)$	Pöschl–Teller II
$C(1-z^2)^2$ (PII)	$-A(A+1)\text{sech}^2(x) + 2B\tanh(x)$	$(-\infty, \infty)$	Rosen–Morse II
	$A(A-1)\text{cosech}^2(x) - 2B\coth(x)$	$[0, \infty)$	Eckart
	$A(A+1)\text{cosec}^2(x) - 2B\cot(x)$	$[0, \pi]$	Rosen–Morse I
Cz (LI)	$\frac{1}{4}\omega^2 x^2 + \frac{l(l+1)}{x^2} - (l + \frac{3}{2})\omega$	$[0, \infty)$	3d harmonic oscillator
C (LII)	$\frac{e^4}{4(l+1)^2} - \frac{e^2}{x} + \frac{l(l+1)}{x^2}$	$[0, \infty)$	Coulomb
Cz^2 (LIII)	$A^2 - B(2A+1)\exp(-x) + B^2\exp(-2x)$	$(-\infty, \infty)$	Morse
C (HI)	$-\frac{1}{2}\omega + \frac{1}{4}\omega^2 x^2$	$(-\infty, \infty)$	1d harmonic oscillator

Obtained by selecting certain single terms on the right handside of $E - V(x) = \ldots$

Constructing more general Natanzon-class potentials

Select certain combinations on the right handside of $E - V(x) = \ldots$

$(z')^2 =$	$z(x)$	$F(z)$	$x \in$	Name	Ref.
$C(1-z^2)^2 z^{-1}$	implicit	$P_n^{(\alpha,\beta)}(z)$	$(-\infty,\infty)$	sym. Ginocchio	Ginocchio 1984
$C(1-z^2)^2 z^{-1}$	implicit	$P_n^{(\alpha,\beta)}(z)$	$[0,\infty)$	PIII	Lévai 1991
$C(z+\theta)$	implicit	$L_n^{(\alpha)}(z)$	$[0,\infty)$	gen. Coulomb	Lévai et al. 1993, 1998
$C(1-z^2)(1-z)z^{-1}$	implicit	$P_n^{(\alpha,\beta)}(z)$	$[0,\infty)$	WRL95	Williams et al. 1995
			confined	WRL95	Williams et al. 1995
$C(1-z^2)^2 z^{-2}$	**explicit**	$P_n^{(\alpha,\beta)}(z)$	$(-\infty,\infty)$	DKV (PIV)	Dutt et al. 1995
$C(1-z^2)^2(z+\gamma)^2$	implicit	$P_n^{(\alpha,\beta)}(z)$	$[0,\infty)$	WL03	Williams et al. 2003
$C(1-z^2)^2(\delta+1-z)^{-1}$	implicit	$P_n^{(\alpha,\beta)}(z)$	$(-\infty,\infty)$	L12	Lévai 2012

Most of these potentials are weakly singular at the finite boundaries

3. Potentials beyond the Natanzon class and the Heun type differential equations

Example 1: the sextic oscillator as a QES potential *Turbiner, Ushveridze*

$$V(r) = \frac{(2s-1/2)(2s-3/2)}{r^2} + \left(b^2 - 4a(s+M+1/2)\right) r^2 + 2abr^4 + a^2r^2$$

Solutions:

$$\psi(r) = Cr^{2s-1/2} \exp\left(-\frac{ar^4}{4} - \frac{br^2}{2}\right) P_M(r^2) \ .$$

$P_M(r^2)$: M'th order polynomial

Its coefficients are determined by a three-term recurrence relation...

...which is terminated by a specific choice of the parameters

This restricts the generality of the potential

The general potential is solved in terms of an infinite power series.

There are infinite number of physical solutions, of which the first $M+1$ can be obtained, hence QES.

The energy eigenvaues are obtained by solving an algebraic equation of order $M+1$.

Example 2: the rationally extended harmonic oscilator potential

Solved by the X_1 type exceptional Laguerre polynomials *Gomez-Ullate et al. 2010*

$$\hat{L}_n^{(\alpha)}(z) = -(\alpha + 1 + z) L_{n-1}^{(\alpha)}(z) + L_{n-2}^{(\alpha)}(z).$$

They satisfy a differential equation similar to generalized Laguerre polynomials

They form an orthogonal basis, but start with degree $\nu = n + 1 > 0$

Linear combinations of two generalized Laguerre polynomials

The potential: *Quesne 2008, Bagchi et al. 2009*

$$\begin{aligned} \hat{V}(r) &= \frac{\omega^2}{4} r^2 + \frac{l(l+1)}{r^2} \\ &\quad + \frac{4\omega}{2l + 1 + \omega r^2} - \frac{8\omega(2l+1)}{(2l+1+\omega r^2)^2}. \end{aligned}$$

Bound-state eigenfunctions:

$$\hat{\psi}_n(l; r) = \hat{C}_n \frac{r^{l+1}}{2l + 1 + \omega r^2} \exp(-\frac{\omega^2}{4} r^2) \hat{L}_{n+1}^{(l+\frac{1}{2})}(\frac{\omega}{2} r^2) \ .$$

Energy eigenvalues:

$$\hat{E}_n = \omega(2n + l + \frac{3}{2}) \ .$$

This potential is related to the harmonic oscillator potential by SUSYQM

It is shape-invariant

Take the harmonic oscillator potential and a SUSY transformation with

$$\chi(r) = r^{l+1} \exp(\frac{\omega^2}{4} r^2)(p + \omega r^2) \ .$$

and factorization energy

$$\epsilon = -\omega(l + \frac{7}{2}) < E_0^{(-)} \ .$$

$V_+(x)$ becomes the rationally extended harmonic oscillator potential

The two spectra are identical: broken SUSY

Alternative approach to these beyond-Natanzon potentials in terms of Heun-type equations

These differential equations and their solutions are more general

The potentials contain more parameters and more (4) terms
They contain the (confluent) hypergeometric case, i.e. the Natanzon potentials

Problem: they are much less elaboreted mathematically

Type	$Q(z)$	$R(z)$	$\exp\left(\frac{1}{2}\int^z Q(z)\mathrm{d}z\right)$
Heun	$\frac{\gamma}{z-a_1}+\frac{\delta}{z-a_2}+\frac{\epsilon}{z-a_3}$	$\frac{\alpha\beta z-q}{(z-a_1)(z-a_2)(z-a_3)}$	$(z-a_1)^{\gamma/2}(z-a_2)^{\delta/2}(z-a_3)^{\epsilon/2}$
Confluent Heun	$4p+\frac{\gamma}{z}+\frac{\delta}{z-1}$	$\frac{4p\beta-\sigma}{(z-1)}+\frac{\sigma}{z}$	$z^{\gamma/2}(z-1)^{\delta/2}\exp(2pz)$
Bi-confluent Heun	$\frac{\gamma}{z}+\delta+\epsilon z$	$\alpha-\frac{q}{z}$	$z^{\gamma/2}\exp\left(\delta z/2+\epsilon z^2/4\right)$
Doubly confluent Heun	$\frac{\delta}{z^2}+\frac{\gamma}{z}+1$	$\frac{\alpha}{z}-\frac{q}{z^2}$	$z^{\gamma/2}\exp\left(-\delta/(2z)+z/2\right)$
Triple confluent Heun	$\gamma z+z^2$	$\alpha z-q$	$\exp\left(\gamma z^2/4+z^3/6\right)$
Hypergeometric (CH)	$\frac{c}{z}+\frac{a+b+1-c}{z-1}$	$\frac{ab}{(z-1)}-\frac{ab}{z}$	$z^{c/2}(z-1)^{(a+b+1-c)/2}$
Confluent hypergeometric (CH, BCH, DCH)	$\frac{b}{z}-1$	$-\frac{a}{z}$	$z^{b/2}\exp(-z/2)$

Apply the method to the bi-confluent Heun equation

$$
\begin{aligned}
E - V(x) &= \frac{z'''(x)}{2z'(x)} - \frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2 \\
&\quad + (z'(x))^2\left[-\frac{\gamma}{2}\left(\frac{\gamma}{2}-1\right)\frac{1}{z^2} - \left(q+\frac{\gamma\delta}{2}\right)\frac{1}{z}\right. \\
&\quad \left. + \left(\alpha - \frac{\epsilon}{2} - \frac{\delta^2}{4} - \frac{\gamma\epsilon}{2}\right) - \frac{\delta\epsilon}{2}z - \frac{\epsilon^2}{4}z^2\right]
\end{aligned}
$$

Define z with

$$
\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 \Phi(z(x)) = C \ ,
$$

where

$$
\Phi(z(x)) = p_1\frac{1}{z^2(x)} + p_2\frac{1}{z(x)} + p_3 + p_4 z(x) + p_5 z^2(x) \ .
$$

Then the pre-factor of the solutions is

$$
f(x) \sim \Phi^{1/4}(z(x))(z(x))^{\gamma/2}\exp\left(\frac{\delta}{2}z(x) + \frac{\epsilon}{4}z^2\right)
$$

The potential is

$$
V(x) = -\frac{z'''(x)}{2z'(x)} + \frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2 + \frac{C}{\Phi(z(x))}\left[s_1\frac{1}{z^2(x)} + s_2\frac{1}{z(x)} + s_3 + s_4 z(x) + s_5 z^2(x)\right].
$$

The parameters and E are related by

$$
\begin{aligned}
&s_1 - p_1 \frac{E}{C} - \frac{\gamma}{2}\left(\frac{\gamma}{2} - 1\right) = 0 \ , \\
&s_2 - p_2 \frac{E}{C} - \left(q + \frac{\gamma\delta}{2}\right) = 0 \ , \\
&s_3 - p_3 \frac{E}{C} + \left(\alpha - \frac{\epsilon}{2} - \frac{\delta^2}{4} - \frac{\gamma\epsilon}{2}\right) = 0 \ , \\
&s_4 - p_4 \frac{E}{C} - \frac{\delta\epsilon}{2} = 0 \ , \\
&s_5 - p_5 \frac{E}{C} - \frac{\epsilon^2}{4} = 0 \ .
\end{aligned}
$$

Now take the following substitutions: $p_2 = 1, \quad p_i = 0, \, i \neq 2$

Then $\Phi(z) = 1/z \quad \longrightarrow z(r) = -\frac{C}{4}x^2$

Take also $\gamma = 2s \quad \delta = -4b/C \quad \epsilon = -16a/C^2 \quad \alpha = 16aM \quad C = 4$

Then

$$V(x) = \frac{(2s - 1/2)(2s - 3/2)}{x^2} + \left(b^2 - 4a(s + M + 1/2)\right) x^2 + 2abr^4 + a^2x^2$$

$$E = 4bs - 4q$$

Polynomial solution of the bi-confluent Heun equation.

Ishkhanyan, Lévai Phys. Scr. 95 (2020) 085202

Apply the method to the confluent Heun equation

$$
\begin{aligned}
E-V(x) &= \frac{z'''(x)}{2z'(x)}-\frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2 \\
&+(z'(x))^2\left(-\frac{\alpha^2}{4}+\frac{2\sigma-\alpha\gamma+\delta\gamma}{2z(x)}+\frac{2\alpha\beta-2\sigma-\alpha\delta-\gamma\delta}{2(z(x)-1)}\right. \\
&\left.+\frac{\gamma(2-\gamma)}{4z^2(x)}+\frac{\delta(2-\delta)}{4(z(x)-1)^2}\right)
\end{aligned}
$$

Define z with

$$
\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2\Phi(z(x))\equiv\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2\frac{\phi(z(x))}{(z(x)(z(x)-1))^2}=C\ ,
$$

where

$$
\phi(z(x))=p_1z^2(x)(z(x)-1)^2+p_2z(x)(z(x)-1)^2+p_3z^2(x)(z(x)-1)+p_4(z(x)-1)^2+p_5z^2(x)\ .
$$

Then the pre-factor of the solutions is

$$
f(x)\sim\phi^{1/4}\exp\left(\frac{\alpha}{2}z(x)\right)(z(x))^{(\gamma-1)/2}(z(x)-1)^{(\delta-1)/2}
$$

The potential is

$$
\begin{aligned}
V(x) &= -\frac{z'''(x)}{2z'(x)}+\frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2+\frac{C}{\phi(z(x))}\Big[s_1z^2(x)(z(x)-1)^2+s_2z(x)(z(x)-1)^2 \\
&+s_3z^2(x)(z(x)-1)+s_4(z(x)-1)^2+s_5z^2(x)\Big]\ .
\end{aligned}
$$

The parameters and E are related by

$$
\begin{aligned}
s_1 - p_1\frac{E}{C} - \frac{\alpha^2}{4} &= 0 \ , \\
s_2 - p_2\frac{E}{C} + \sigma - \frac{\alpha\gamma}{2} + \frac{\delta\gamma}{2} &= 0 \ , \\
s_3 - p_3\frac{E}{C} + \alpha\beta - \sigma - \frac{\alpha\delta}{2} - \frac{\gamma\delta}{2} &= 0 \ , \\
s_4 - p_4\frac{E}{C} - \frac{\gamma}{2}\left(\frac{\gamma}{2} - 1\right) &= 0 \ , \\
s_5 - p_5\frac{E}{C} - \frac{\delta}{2}\left(\frac{\delta}{2} - 1\right) &= 0 \ .
\end{aligned}
$$

Now take the following substitutions: $p_2 = 1, \quad p_i = 0,\ i \neq 2$

Then $\phi(z) = z(z-1)^2 \quad \longrightarrow z(r) = -\frac{C}{4} r^2$

Take also $\beta = -N \quad \gamma = \alpha + 1 \quad \delta = -2 \quad \sigma = (2-N)\alpha$

$C = -2\omega/2 \quad \alpha = l + 1/2 \quad s_2 = 1$

Then

$$V(x) = \frac{\omega^2}{4} x^4 + \frac{l(l+1)}{x^2} + \frac{4\omega}{\omega x^2 + 2l + 1} - \frac{8\omega(2l+1)}{(\omega x^2 + 2l + 1)^2} \ ,$$

$$E_N = \omega(2N + l - 1/2)$$

This is a polynomial solution of the confluent Heun equation.

The CHE reduces to that of the X_1 type exceptional Laguerre polynomials

$$\psi_N(x) \sim \frac{x^{l+1}}{\omega x^2 + 2l + 1} \exp\left(-\frac{\omega}{4} x^2\right) \hat{L}_N^{(l+1/2)}\left(\frac{\omega}{2} x^2\right) \ .$$

Potentials related to the X_1 type exceptional Jacobi polynomials also follow from the CHE

Discussion

Variable transforations and SUSYQM help a lot to find solvable potetials

They also help in the classification of solvable potentials

Natanzon-class potentials: $F(z)$ is the (confluent) hypergeometric funcion

– Bound states are described by classical orthogonal polynomials

Potentials beyond the Natanzon class?

– They have been found using different methods

QES

Exceptional orthogonal polynomials

They can be discussed in a united for in terms of Heun-type equations

– Polynomial solutions of the CHE: rationally extended harmonic oscillator

– Polynomial solutions of the CHE: rationally extended Scarf II potential (not shown)

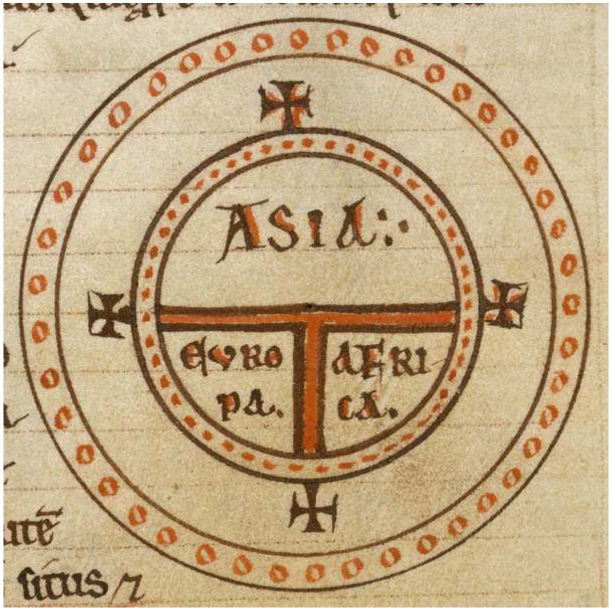
– Polynomial solutions of the BHE: sextic oscillator

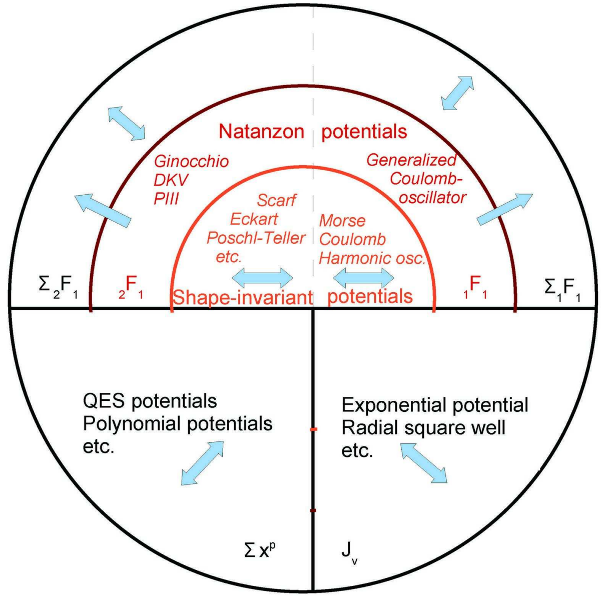
The solutions arise as the combination of two ordinary polynomials (recursion!)

This paves the way to SUSYQM: polynomial + its derivative

Natanzon-clss potentials included as special cases

The world map of solvable potentials revisited





We found the legendary islands