

Propagating-wave approximation and the transfer matrix in two-dimensional potential scattering



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Joint Work with
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Outline:

- Potential scattering in 1D & the transfer matrix
- Potential scattering in 2D & its dynamical formulation
- Evanescent waves in 2D & the Propagating-wave approximation (PWA)
- Nonlocal potentials V_k achieving PWA
- Potentials for which PWA is exact
- Existence of the transfer matrix for V_k
- Concluding remarks

Scattering in 1D

$$\Psi(x, t) = e^{-i\omega t} \psi(x)$$

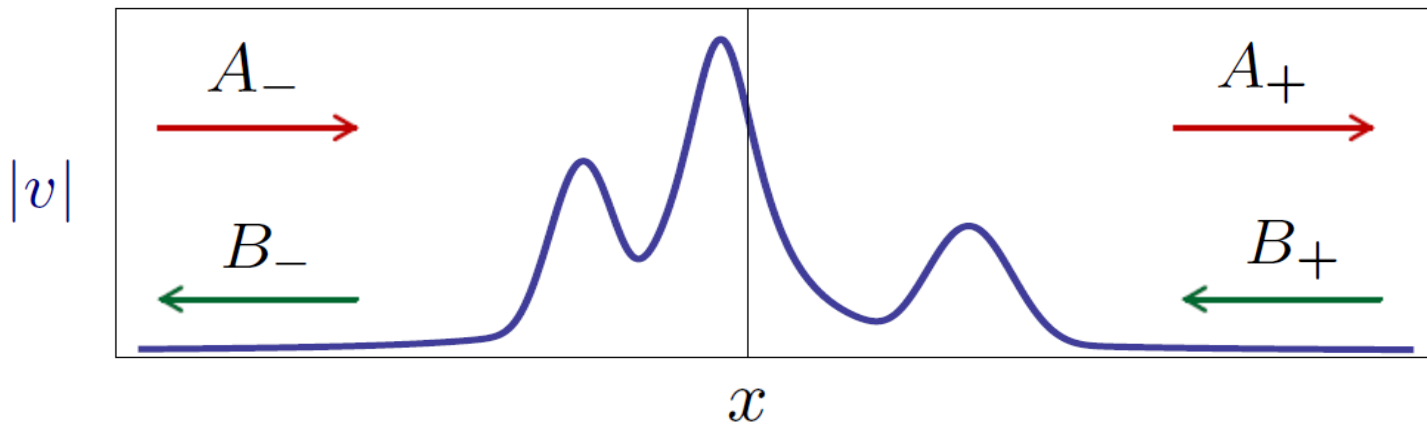
- Time-Indep. Schrödinger Eq.: $-\psi(x)'' + v(x)\psi(x) = k^2\psi(x)$
- $v : \mathbb{R} \rightarrow \mathbb{C}$ is a possibly k -dependent **short-range potential**.

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$$\psi(x) \rightarrow \begin{cases} A_- e^{ikx} + B_- e^{-ikx} & \text{for } x \rightarrow -\infty \\ A_+ e^{ikx} + B_+ e^{-ikx} & \text{for } x \rightarrow +\infty \end{cases}$$

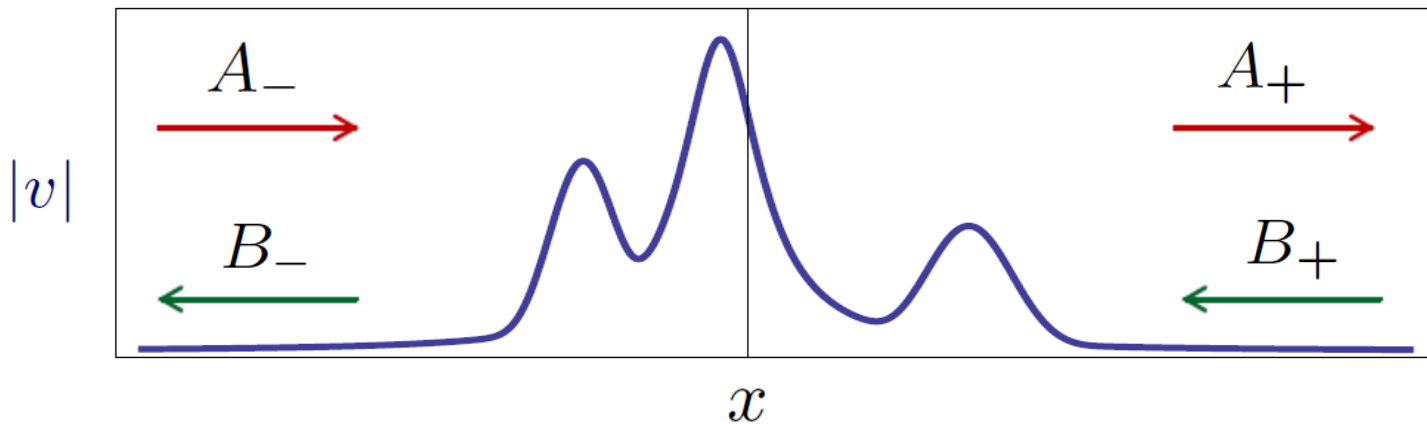


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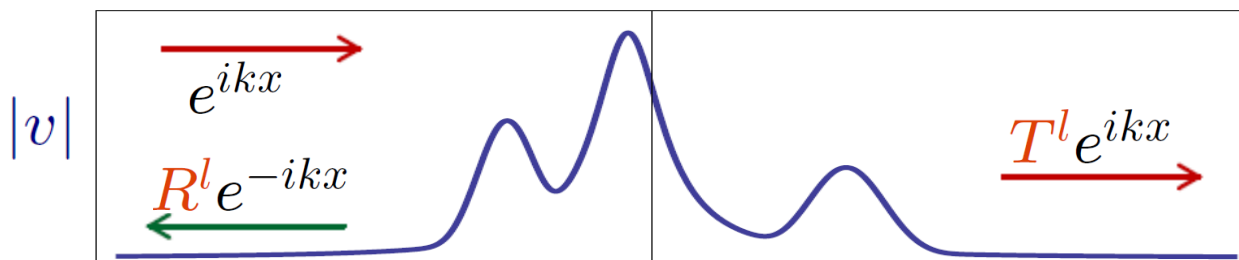
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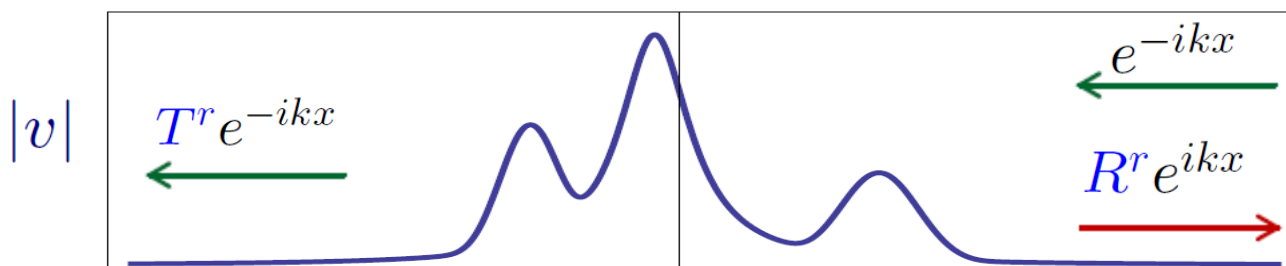
- Transfer matrix: $\begin{bmatrix} A_+ \\ B_+ \end{bmatrix} = \mathbf{M} \begin{bmatrix} A_- \\ B_- \end{bmatrix}.$

- Scattering from the left and right:

$$\psi^{\text{left}}(x) = \begin{cases} e^{ikx} + R^l e^{-ikx} & \text{for } x \rightarrow -\infty \\ T^l e^{ikx} & \text{for } x \rightarrow +\infty \end{cases}$$

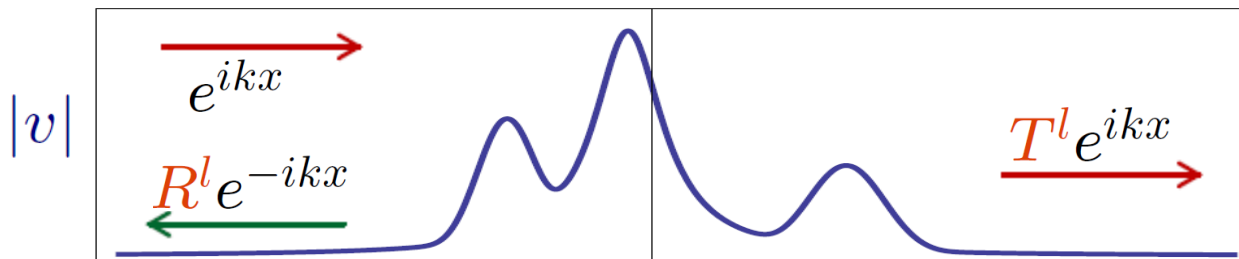


$$\psi^{\text{right}}(x) = \begin{cases} T^r e^{-ikx} & \text{for } x \rightarrow -\infty \\ e^{-ikx} + R^r e^{ikx} & \text{for } x \rightarrow +\infty \end{cases}$$

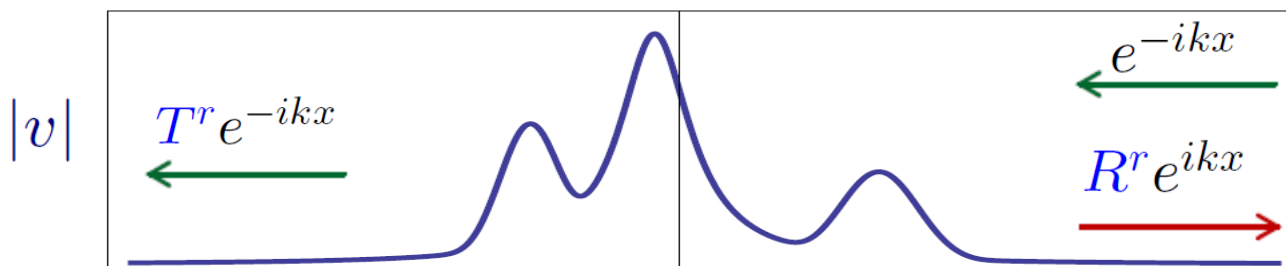


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$$R^l = -\frac{M_{21}}{M_{22}}, \quad R^r = \frac{M_{12}}{M_{22}}, \quad T^l = T^r =: T = \frac{1}{M_{22}}.$$

Composition Property of M

Let v_1 and v_2 be scattering potentials such that

$$v_1(x) = 0 \quad \text{for} \quad x > a,$$

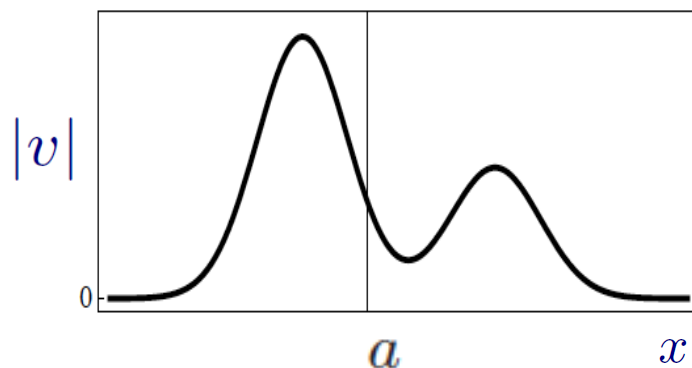
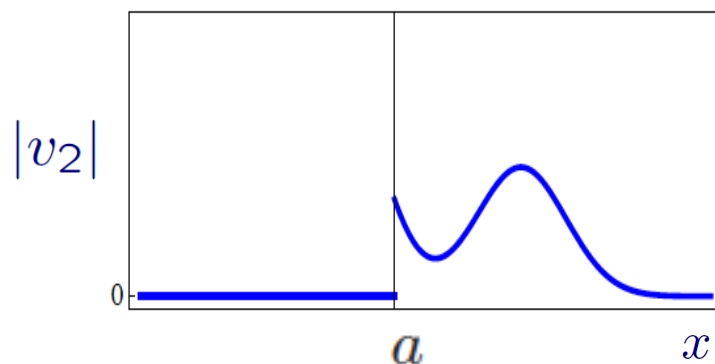
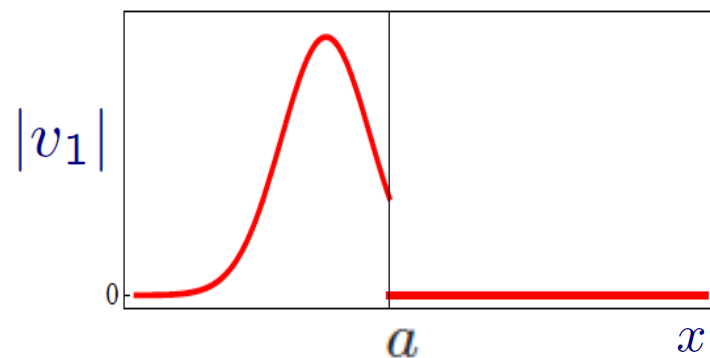
$$v_2(x) = 0 \quad \text{for} \quad x < a$$

$$v(x) = v_1(x) + v_2(x).$$

\mathbf{M}_1 : Transfer matrix of v_1

\mathbf{M}_2 : Transfer matrix of v_2

\mathbf{M} : Transfer matrix of $v = v_1 + v_2$



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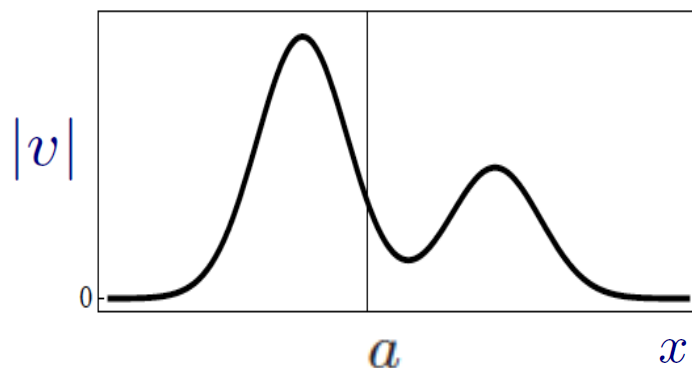
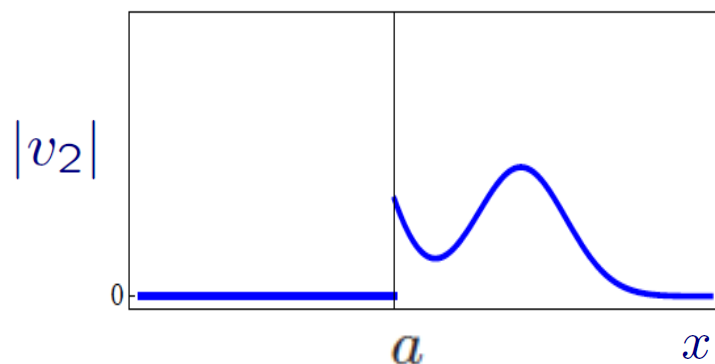
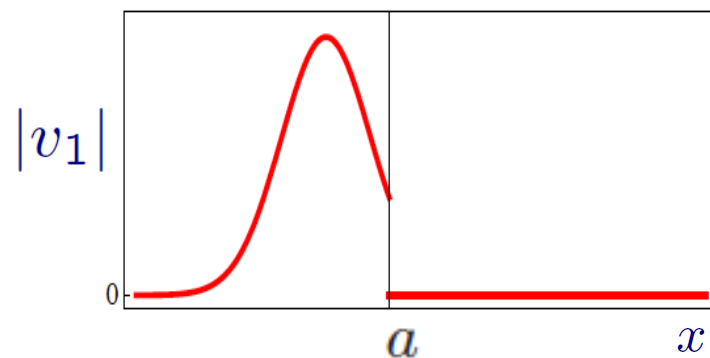
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M_1 : Transfer matrix of v_1

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M : Transfer matrix of $v = v_1 + v_2$

Then $M = M_2 M_1$.



Theorem: $\mathbf{M} = \mathbf{U}(+\infty, -\infty)$ where $\mathbf{U}(x, x_0)$ is the evolution operator for

$$\mathbf{H}(x) := \frac{v(x)}{2k} \begin{bmatrix} 1 & e^{-2ikx} \\ -e^{2ikx} & -1 \end{bmatrix}.$$

x plays the role of “time”.

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$$i\partial_x \mathbf{U}(x, x_0) = \mathbf{H}(x) \mathbf{U}(x, x_0),$$

$$\mathbf{U}(x_0, x_0) = \mathbf{I}.$$

[Ann. Phys. (NY), **341**, 77 (2014)]

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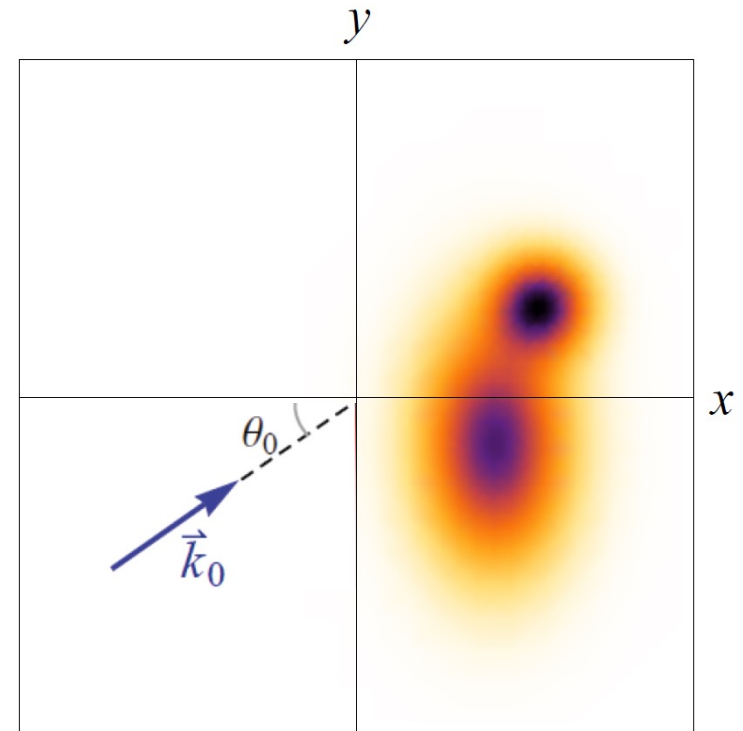
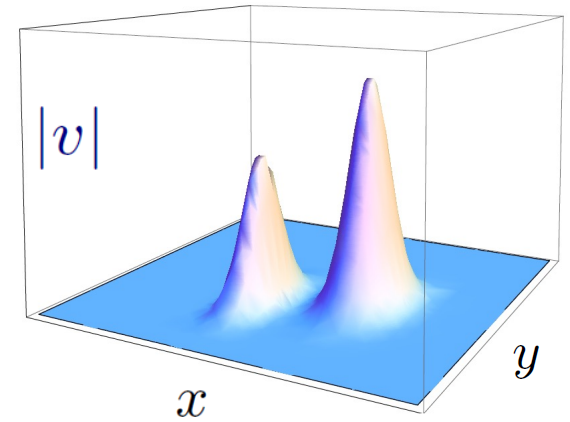
i.e.,

$$\begin{aligned} \mathbf{M} &= \mathcal{T} \exp \int_{-\infty}^{\infty} -i\mathbf{H}(x)dx \\ &= \mathbf{I} - i \int_{-\infty}^{\infty} dx_1 \mathbf{H}(x_1) \\ &\quad + (-i)^2 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{x_2} dx_1 \mathbf{H}(x_2) \mathbf{H}(x_1) + \cdots \end{aligned}$$

[Ann. Phys. (NY), **341**, 77 (2014)]

Scattering in 2D

$$[-\partial_x^2 - \partial_y^2 + v(x, y)] \psi(x, y) = k^2 \psi(x, y)$$



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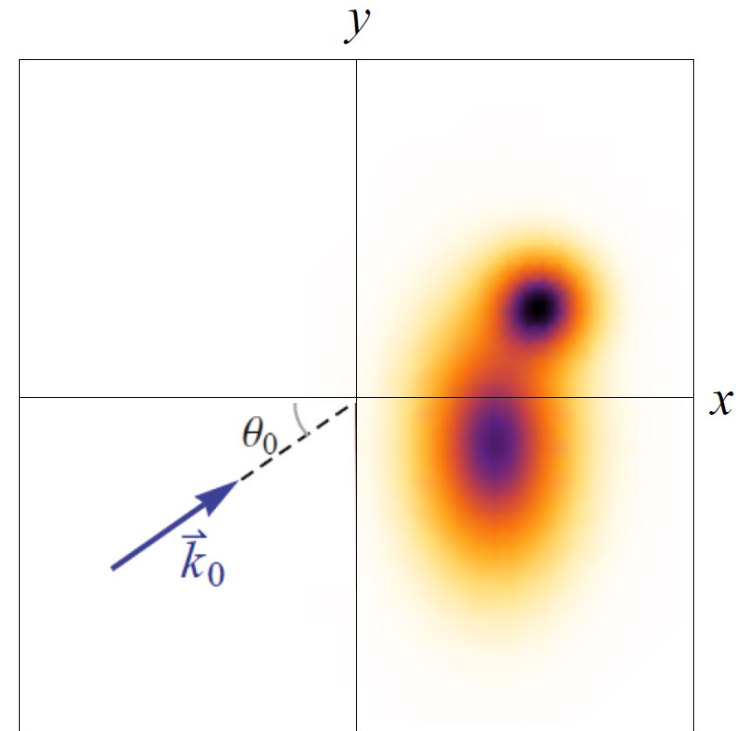
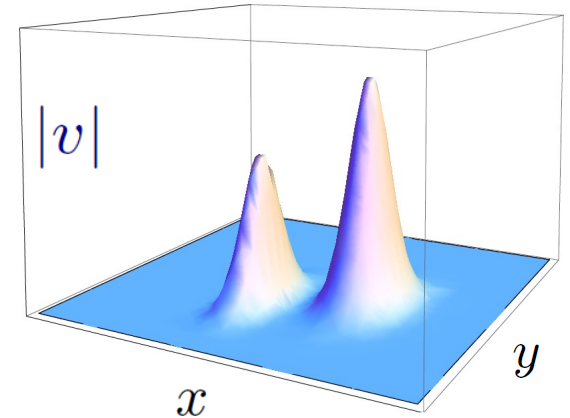
$$\psi(\mathbf{r}) \rightarrow \frac{e^{i\mathbf{k}_0 \cdot \mathbf{r}}}{2\pi} + \sqrt{\frac{i}{kr}} \frac{e^{ikr} f(\theta)}{2\pi} \quad \text{as } r \rightarrow \infty$$

\mathbf{k}_0 := Incident wave vector

$\mathbf{r} := (x, y)$

$(r, \theta) :=$ polar coordinates

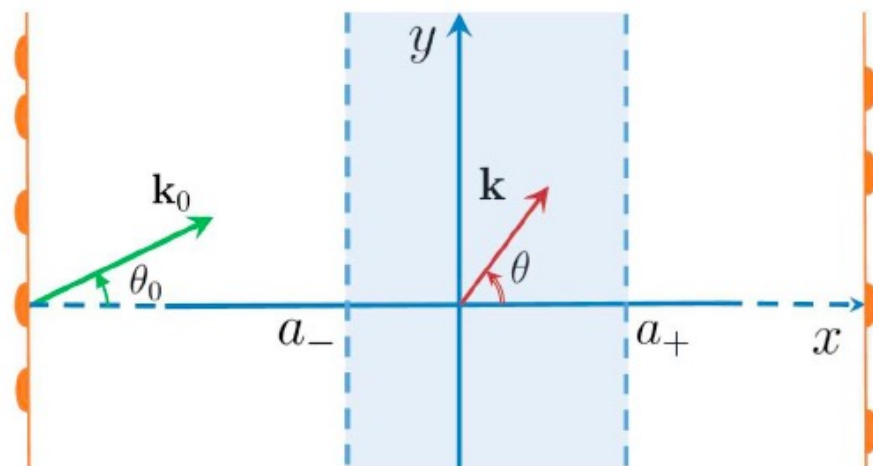
$f(\theta)$: Scattering amplitude



Suppose that $v(x, y) = 0$ for $x \notin [a_-, a_+]$

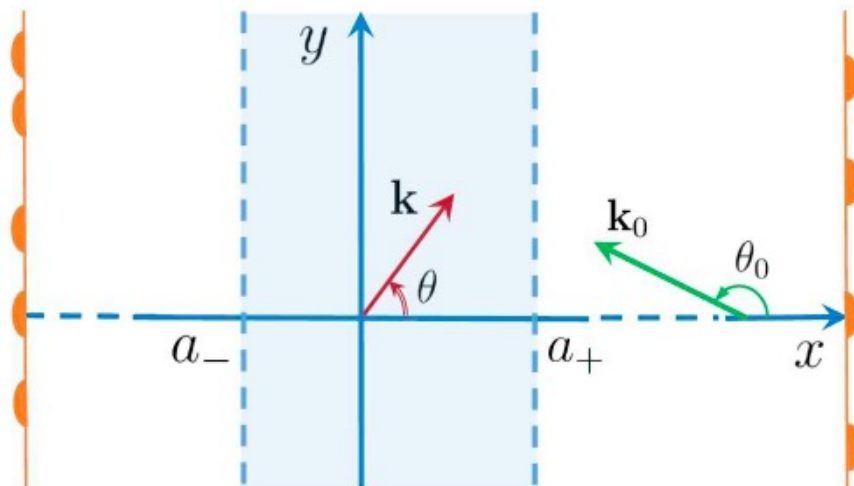
Suppose that $v(x, y) = 0$ for $x \notin [a_-, a_+]$

Left-incident wave



$\mathbf{k} := \frac{k \mathbf{r}}{r}$: scattered wave vector

Right-incident wave



In regions where $v(x, y) = 0$,

$$(-\nabla^2 + v)\psi = k^2\psi \quad \rightarrow \quad (\nabla^2 + k^2)\psi = 0$$

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Plane-wave solutions:

$$e^{\pm i\sqrt{k^2 - p^2}x} e^{ipy} \quad \text{with} \quad p \in (-k, k)$$

Evanescent-wave solutions:

$$e^{\pm \sqrt{p^2 - k^2}x} e^{ipy} \quad \text{with} \quad p \notin (-k, k)$$

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$$e^{\pm i\varpi(p)x} e^{ipy} \quad \text{with} \quad p \in \mathbb{R},$$

$$\varpi(p) := \begin{cases} \sqrt{k^2 - p^2} & \text{for } |p| < k, \\ i\sqrt{p^2 - k^2} & \text{for } |p| \geq k. \end{cases}$$

Bounded solutions: $\psi = \psi_{\text{os}} + \psi_{\text{ev}}$

$$\psi_{\text{os}}(x, y) = \begin{cases} \int_{-k}^k \frac{dp e^{ipy}}{4\pi^2 \varpi(p)} [A_{-}(p) e^{i\varpi(p)x} + B_{-}(p) e^{-i\varpi(p)x}] & \text{for } x \leq a_{-}, \\ \int_{-k}^k \frac{dp e^{ipy}}{4\pi^2 \varpi(p)} [A_{+}(p) e^{i\varpi(p)x} + B_{+}(p) e^{-i\varpi(p)x}] & \text{for } x \geq a_{+}, \end{cases}$$

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$\mathcal{F} :=$ Set of complex-valued functions of p ,

$$\mathcal{F}_k := \{ \phi \in \mathcal{F} \mid \phi(p) = 0 \text{ for } p \notin (-k, k) \}$$

$$A_{\pm}, B_{\pm} \in \mathcal{F}_k$$

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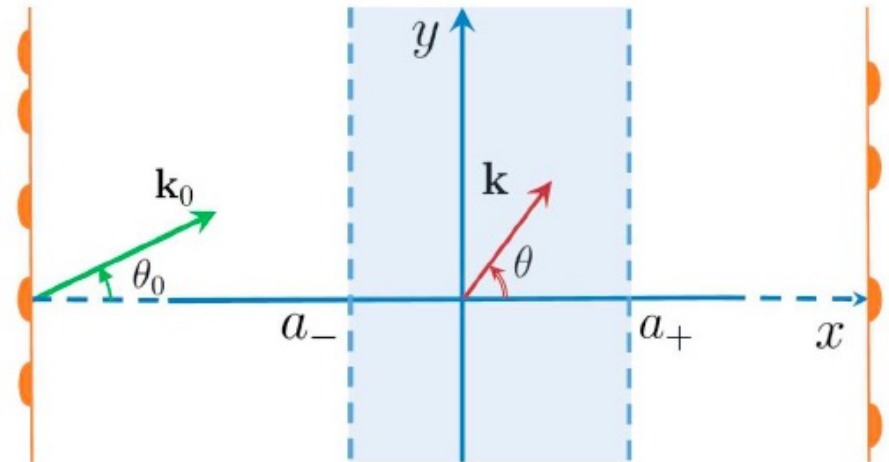
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Fundamental transfer matrix: $\widehat{\mathbf{M}} : \mathcal{F}_k^2 \rightarrow \mathcal{F}_k^2,$

$$\widehat{\mathbf{M}} \begin{bmatrix} A_{-} \\ B_{-} \end{bmatrix} = \begin{bmatrix} A_{+} \\ B_{+} \end{bmatrix}$$

$$\psi(x, y) \rightarrow \int_{-k}^k \frac{dp e^{ipy}}{4\pi^2 \varpi(p)} \left[A_{\pm}(p) e^{i\varpi(p)x} + B_{\pm}(p) e^{-i\varpi(p)x} \right] \text{ for } x \rightarrow \pm\infty$$

For left-incident waves: $\mathbf{k}_0 = (\varpi(p_0), p_0)$, $p_0 \in (-k, k)$

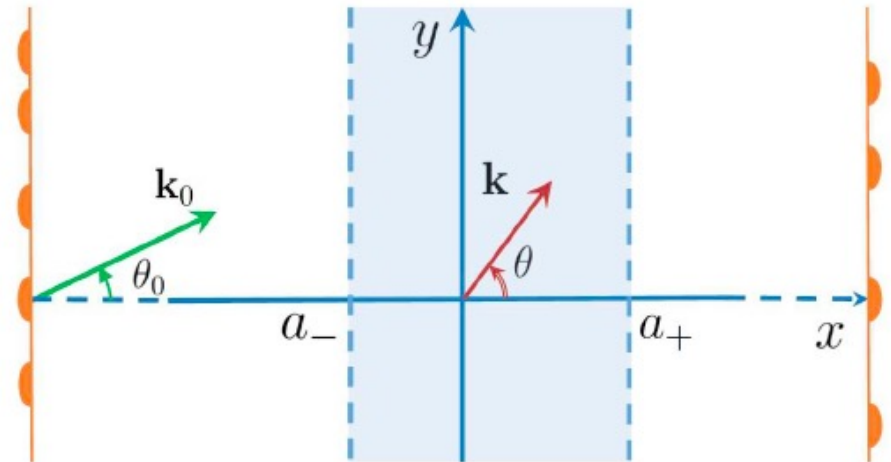


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$$\begin{aligned} A_{-}(p) &= 2\pi \varpi(p_0) \delta(p - p_0) \\ &= 2\pi \delta(\theta - \theta_0), \end{aligned}$$

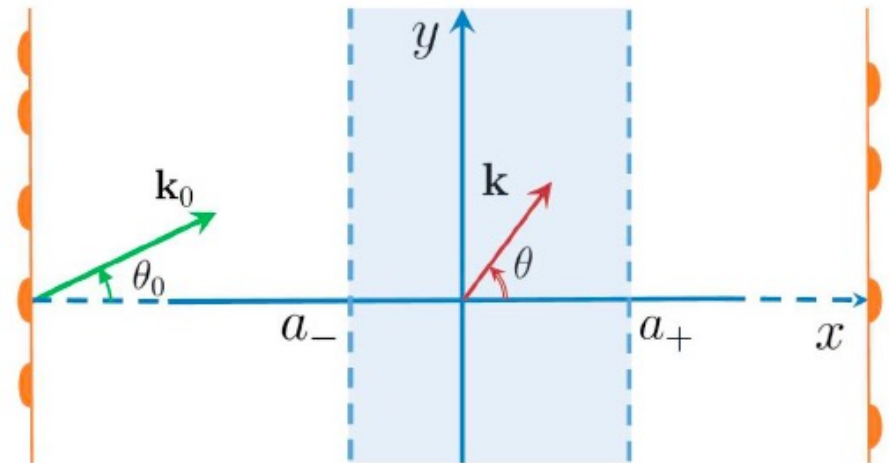
$$B_{+}(p) = 0.$$



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For left-incident waves: $\mathbf{k}_0 = (\varpi(p_0), p_0)$, $p_0 \in (-k, k)$

$$f(\theta) = -\frac{i}{\sqrt{2\pi}} \times \begin{cases} A_+(k \sin \theta) - 2\pi \delta(\theta - \theta_0) & \text{for } \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}), \\ B_-(k \sin \theta) & \text{for } \theta \in (\frac{\pi}{2}, \frac{3\pi}{2}), \end{cases}$$



[PRA **93**, 042707 (2016) & **104**, 032222 (2021)]

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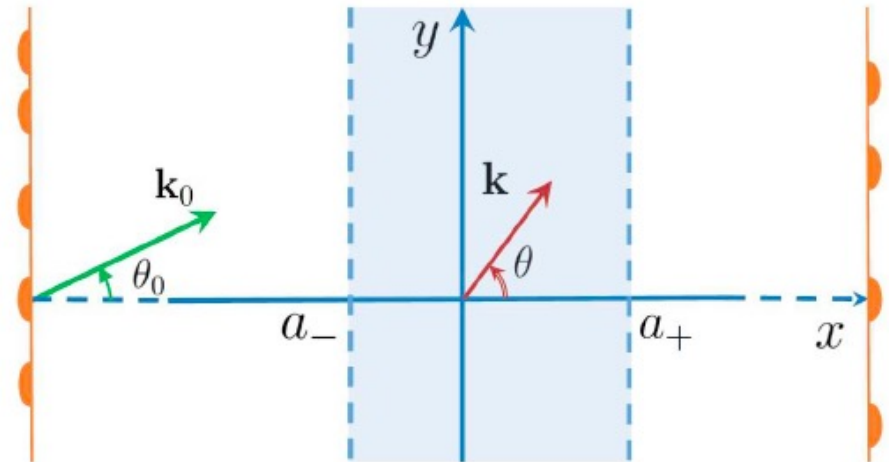
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$$\widehat{M}_{22} B_- = -2\pi \widehat{M}_{21} \delta_{p_0},$$

$$A_+ = 2\pi \widehat{M}_{11} \delta_{p_0} + \widehat{M}_{12} B_-$$

$$\delta_{p_0}(p) := \delta(p - p_0)$$



[PRA **93**, 042707 (2016) & **104**, 032222 (2021)]

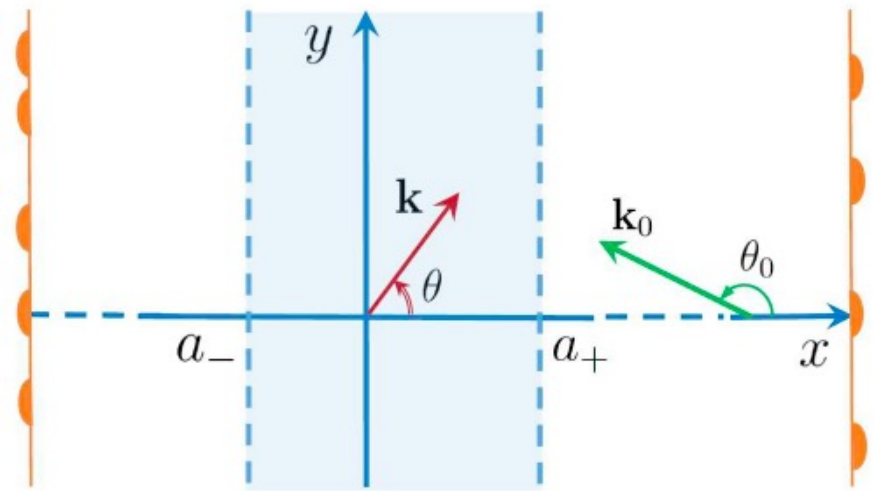
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For right-incident waves: $\mathbf{k}_0 = (-\varpi(p_0), p_0)$, $p_0 \in (-k, k)$

$$f(\theta) = \frac{i}{\sqrt{2\pi}} \times \begin{cases} A_+(k \sin \theta) & \text{for } \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}), \\ B_-(k \sin \theta) - 2\pi \delta(\theta - \theta_0) & \text{for } \theta \in (\frac{\pi}{2}, \frac{3\pi}{2}), \end{cases}$$

$$\widehat{M}_{22} B_- = 2\pi \delta_{p_0},$$

$$A_+ = \widehat{M}_{12} B_-$$



Projection onto \mathcal{F}_k^2 : $(\hat{\Pi}_k F)(p) := \begin{cases} F(p) & \text{for } p \in (-k, k) \\ 0 & \text{for } p \notin (-k, k) \end{cases}$

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Theorem: $\hat{\mathbf{M}} = \hat{\Pi}_k \hat{\mathfrak{M}} \hat{\Pi}_k$, where $\hat{\mathfrak{M}} := \hat{\mathcal{U}}(-\infty, \infty)$,

$$\hat{\mathcal{U}}(x, x_0) := \mathcal{T} \exp \left\{ -i \int_{x_0}^x dx' \hat{\mathcal{H}}(x') \right\},$$

Projection onto \mathcal{F}_k^2 : $(\hat{\Pi}_k F)(p) := \begin{cases} F(p) & \text{for } p \in (-k, k) \\ 0 & \text{for } p \notin (-k, k) \end{cases}$

Theorem: $\hat{\mathbf{M}} = \hat{\Pi}_k \hat{\mathfrak{M}} \hat{\Pi}_k$, where $\hat{\mathfrak{M}} := \hat{\mathcal{U}}(-\infty, \infty)$,

$$\hat{\mathcal{U}}(x, x_0) := \mathcal{T} \exp \left\{ -i \int_{x_0}^x dx' \hat{\mathcal{H}}(x') \right\},$$

$$\hat{\mathcal{H}}(x) := \frac{1}{2} e^{-i \hat{\varpi} x \sigma_3} v(x, \hat{y}) \hat{\varpi}^{-1} \kappa e^{i \hat{\varpi} x \sigma_3},$$

$$(\hat{\varpi} f)(p) := \varpi(p) f(p), \quad \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\varpi(p) := \begin{cases} \sqrt{k^2 - p^2} & \text{for } |p| < k, \\ i \sqrt{p^2 - k^2} & \text{for } |p| \geq k, \end{cases}$$

$$(\hat{y} f)(p) := i \partial_p f(p), \quad \kappa := \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

Projection onto \mathcal{F}_k^2 : $(\hat{\Pi}_k F)(p) := \begin{cases} F(p) & \text{for } p \in (-k, k) \\ 0 & \text{for } p \notin (-k, k) \end{cases}$

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$$(v(x, \hat{y}) f)(p) := \frac{1}{2\pi} \int_{-\infty}^{\infty} dq \tilde{v}(x, p - q) f(q),$$

$$\tilde{v}(x, p) := \int_{-\infty}^{\infty} dy e^{-ipy} v(x, y).$$

$$\begin{aligned}
\widehat{\mathbf{M}} &:= \widehat{\Pi}_k \mathcal{T} \exp \left\{ -i \int_{-\infty}^{\infty} dx \widehat{\mathcal{H}}(x) \right\} \widehat{\Pi}_k \\
&= \widehat{\mathbf{I}} - i \int_{-\infty}^{\infty} dx_1 \widehat{\Pi}_k \widehat{\mathcal{H}}(x_1) \widehat{\Pi}_k \\
&\quad - \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{x_2} dx_1 \widehat{\Pi}_k \widehat{\mathcal{H}}(x_2) \widehat{\mathcal{H}}(x_1) \widehat{\Pi}_k + \cdots
\end{aligned}$$

$$(\widehat{\Pi}_k F)(p) := \begin{cases} F(p) & \text{for } p \in (-k, k) \\ 0 & \text{for } p \notin (-k, k) \end{cases}$$

$$\widehat{\mathcal{H}}(x) := \frac{1}{2} e^{-i\widehat{\varpi}x\sigma_3} v(x, \widehat{y}) \widehat{\varpi}^{-1} \mathcal{K} e^{i\widehat{\varpi}x\sigma_3},$$

$$\mathcal{K} := \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Hybrid Rep:

For each $x \in \mathbb{R}$, $|\psi(x)\rangle : \mathbb{R} \rightarrow \mathbb{C}$ is the function:

$$\langle y|\psi(x)\rangle := \psi(x, y)$$

$$[-\partial_x^2 + \hat{p}^2 + v(x, \hat{y})]|\psi(x)\rangle = k^2|\psi(x)\rangle$$

$$\langle y|\hat{y}|\phi\rangle = y\langle y|\phi\rangle, \quad \langle y|\hat{p}|\phi\rangle := -i\partial_y\langle y|\phi\rangle$$

$$\hat{\Pi}_k := \int_{-k}^k dp |p\rangle\langle p|, \quad \langle y|p\rangle = \frac{e^{ipy}}{\sqrt{2\pi}}$$

$$\langle p|\hat{\Pi}_k|\phi\rangle = \begin{cases} \langle p|\phi\rangle & \text{for } p \in (-k, k) \\ 0 & \text{for } p \notin (-k, k) \end{cases}$$

$$\hat{\Pi}_k := \int_{-k}^k dp |p\rangle \langle p|, \quad \langle p | \hat{\Pi}_k | \phi \rangle = \begin{cases} \langle p | \phi \rangle & \text{for } p \in (-k, k) \\ 0 & \text{for } p \notin (-k, k) \end{cases}$$

$$\psi = \psi_{\text{os}} + \psi_{\text{ev}}$$

$$\psi_{\text{os}}(x, y) = \begin{cases} \int_{-k}^k \frac{dp e^{ipy}}{4\pi^2 \varpi(p)} [A_-(p) e^{i\varpi(p)x} + B_-(p) e^{-i\varpi(p)x}] & \text{for } x \leq a_-, \\ \int_{-k}^k \frac{dp e^{ipy}}{4\pi^2 \varpi(p)} [A_+(p) e^{i\varpi(p)x} + B_+(p) e^{-i\varpi(p)x}] & \text{for } x \geq a_+, \end{cases}$$

$$\psi_{\text{ev}}(x, y) = \begin{cases} \int_{|p| \geq k} \frac{dp e^{ipy}}{4\pi^2 \varpi(p)} C_-(p) e^{|\varpi(p)|x} & \text{for } x \leq a_-, \\ \int_{|p| \geq k} \frac{dp e^{ipy}}{4\pi^2 \varpi(p)} C_+(p) e^{-|\varpi(p)|x} & \text{for } x \geq a_+, \end{cases}$$

$$A_{\pm}(p) = B_{\pm}(p) = 0 \quad \text{for } |p| \geq k, \quad C_{\pm}(p) = 0 \quad \text{for } |p| < k.$$

$$\hat{\Pi}_k |A_{\pm}\rangle = |A_{\pm}\rangle, \quad \hat{\Pi}_k |B_{\pm}\rangle = |B_{\pm}\rangle, \quad \hat{\Pi}_k |C_{\pm}\rangle = 0.$$

$$\hat{\Pi}_k := \int_{-k}^k dp |p\rangle \langle p|, \quad \langle p | \hat{\Pi}_k | \phi \rangle = \begin{cases} \langle p | \phi \rangle & \text{for } p \in (-k, k) \\ 0 & \text{for } p \notin (-k, k) \end{cases}$$

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$$\hat{\Pi}_k |A_{\pm}\rangle = |A_{\pm}\rangle, \quad \hat{\Pi}_k |B_{\pm}\rangle = |B_{\pm}\rangle, \quad \hat{\Pi}_k |C_{\pm}\rangle = 0.$$

$$|\psi_{\text{os}}(x)\rangle := \hat{\Pi}_k |\psi(x)\rangle \quad |\psi_{\text{ev}}(x)\rangle := |\psi(x)\rangle - |\psi_{\text{os}}(x)\rangle$$

$$\hat{\Pi}_k := \int_{-k}^k dp |p\rangle\langle p|,$$

$$\psi = \psi_{\text{os}} + \psi_{\text{ev}}$$

$$|\psi_{\text{os}}(x)\rangle := \hat{\Pi}_k |\psi(x)\rangle$$

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Propagating-wave approximation (PWA): $\psi \approx \psi_{\text{os}}$

PWA neglects the contribution of the evanescent waves to the scattering amplitude.

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PWA neglects the contribution of the evanescent waves to the scattering amplitude.

$$[-\partial_x^2 + \hat{p}^2 + v(x, \hat{y})] |\psi(x)\rangle = k^2 |\psi(x)\rangle$$

$$\Rightarrow \left[-\partial_x^2 + \hat{p}^2 + \hat{V}_k(x) \right] |\psi_{\text{os}}(x)\rangle + \hat{W}_k(x) |\psi_{\text{ev}}(x)\rangle = k^2 |\psi_{\text{os}}(x)\rangle,$$

$$\hat{V}_k(x) := \hat{\Pi}_k v(x, \hat{y}) \hat{\Pi}_k, \quad \hat{W}_k(x) := \hat{\Pi}_k v(x, \hat{y}) (\hat{1} - \hat{\Pi}_k).$$

$$\hat{\Pi}_k := \int_{-k}^k dp |p\rangle \langle p|,$$

$$\psi = \psi_{\text{os}} + \psi_{\text{ev}}$$

$$|\psi_{\text{os}}(x)\rangle := \hat{\Pi}_k |\psi(x)\rangle$$

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PWA: $[-\partial_x^2 + \hat{p}^2 + \hat{V}_k(x)] |\psi(x)\rangle = k^2 |\psi(x)\rangle$ i.e., $v \longrightarrow \mathcal{V}_k$

$$(\mathcal{V}_k \psi)(x, y) := \langle y | \hat{V}_k(x) | \psi(x) \rangle$$

[arXiv: 2204.05153]

$$\left[-\partial_x^2 + \hat{p}^2 + \hat{V}_k(x) \right] |\psi_{\text{os}}(x)\rangle + \hat{W}_k(x) |\psi_{\text{ev}}(x)\rangle = k^2 |\psi_{\text{os}}(x)\rangle$$

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$$\mathbf{PWA} : -\nabla^2 \psi(x, y) + (\mathcal{V}_k \psi)(x, y) = k^2 \psi(x, y)$$

$$(\mathcal{V}_k \psi)(x, y) := \langle y | \hat{V}_k(x) | \psi(x) \rangle$$

$$\left[-\partial_x^2 + \hat{p}^2 + \hat{V}_k(x) \right] |\psi_{\text{os}}(x)\rangle + \hat{W}_k(x) |\psi_{\text{ev}}(x)\rangle = k^2 |\psi_{\text{os}}(x)\rangle$$

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$$\textbf{PWA} : -\nabla^2 \psi(x, y) + (\mathcal{V}_k \psi)(x, y) = k^2 \psi(x, y)$$

$$(\mathcal{V}_k \psi)(x, y) := \langle y | \hat{V}_k(x) | \psi(x) \rangle$$

$$= \frac{1}{4\pi^2} \int_{-k}^k dp \int_{-k}^k dq e^{iyp} \tilde{v}(x, p - q) \tilde{\psi}(x, q)$$

$$\tilde{v}(x, p) := \int_{-\infty}^{\infty} dy e^{-ipy} v(x, y)$$

\mathcal{V}_k is an energy-dependent nonlocal potential.

$$\left[-\partial_x^2 + \hat{p}^2 + \hat{V}_k(x) \right] |\psi_{\text{os}}(x)\rangle + \hat{W}_k(x) |\psi_{\text{ev}}(x)\rangle = k^2 |\psi_{\text{os}}(x)\rangle$$

$$\hat{V}_k(x) := \hat{\Pi}_k v(x, \hat{y}) \hat{\Pi}_k, \quad \hat{W}_k(x) := \hat{\Pi}_k v(x, \hat{y}) (\hat{1} - \hat{\Pi}_k).$$

$$\textbf{PWA} : -\nabla^2 \psi(x, y) + (\mathcal{V}_k \psi)(x, y) = k^2 \psi(x, y)$$

$$(\mathcal{V}_k \psi)(x, y) := \langle y | \hat{V}_k(x) | \psi(x) \rangle$$

$$\text{For } k \rightarrow \infty, \hat{\Pi}_k := \int_{-k}^k dp |p\rangle \langle p| \rightarrow \hat{1}$$

$$\Rightarrow \hat{V}_k(x) \rightarrow v(x, \hat{y}) \text{ \& } \hat{W}_k(x) \rightarrow \hat{0}$$

$$\Rightarrow \textbf{PWA} \text{ is valid at high energies.}$$

$$\left[-\partial_x^2 + \hat{p}^2 + \hat{V}_k(x) \right] |\psi_{\text{os}}(x)\rangle + \hat{W}_k(x) |\psi_{\text{ev}}(x)\rangle = k^2 |\psi_{\text{os}}(x)\rangle$$

$$\hat{V}_k(x) := \hat{\Pi}_k v(x, \hat{y}) \hat{\Pi}_k, \quad \hat{W}_k(x) := \hat{\Pi}_k v(x, \hat{y}) (\hat{1} - \hat{\Pi}_k).$$

Theorem: Every short-range potential $v : \mathbb{R}^2 \rightarrow \mathbb{C}$ couples to evanescent waves, i.e., $\hat{W}_k(x) \neq \hat{0}$.

Nevertheless, PWA can produce the exact expression for the scattering amplitudes!

Transfer matrix for \mathcal{V}_k :

Recall: $\widehat{\mathbf{M}} : \mathcal{F}_k^2 \rightarrow \mathcal{F}_k^2$,

$$F \in \mathcal{F}_k^2 \quad \Leftrightarrow \quad F(p) = 0 \quad \text{for } p \notin (-k, k)$$

Transfer matrix for \mathcal{V}_k :

Recall: $\widehat{\mathbf{M}} : \mathcal{F}_k^2 \rightarrow \mathcal{F}_k^2$,

$$\widehat{\mathbf{M}} := \widehat{\Pi}_k \mathcal{T} \exp \left\{ -i \int_{-\infty}^{\infty} dx \widehat{\mathcal{H}}(x) \right\} \widehat{\Pi}_k$$

$$\widehat{\mathcal{H}}(x) := \frac{1}{2} e^{-i \widehat{\varpi} x \sigma_3} v(x, \widehat{y}) \widehat{\varpi}^{-1} \kappa e^{i \widehat{\varpi} x \sigma_3}$$

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$$\text{PWA} : v \longrightarrow \mathcal{V}_k \quad \Leftrightarrow \quad v(x, \widehat{y}) \longrightarrow \widehat{V}_k(x) := \widehat{\Pi}_k v(x, \widehat{y}) \widehat{\Pi}_k$$

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$$\begin{aligned} \text{PWA} : v &\longrightarrow \mathcal{V}_k & \Leftrightarrow & v(x, \widehat{y}) \longrightarrow \widehat{V}_k(x) := \widehat{\Pi}_k v(x, \widehat{y}) \widehat{\Pi}_k \\ & & \Rightarrow & \widehat{\Pi}_k \widehat{V}_k(x) \widehat{\Pi}_k = \widehat{V}_k(x) \end{aligned}$$

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Recall: $\widehat{\mathbf{M}} : \mathcal{F}_k^2 \rightarrow \mathcal{F}_k^2$,

$$\widehat{\mathbf{M}} := \widehat{\Pi}_k \mathcal{T} \exp \left\{ -i \int_{-\infty}^{\infty} dx \widehat{\mathcal{H}}(x) \right\} \widehat{\Pi}_k$$

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$$\widehat{\mathbf{M}} \longrightarrow \widehat{\mathbf{M}}_k := \widehat{\Pi}_k \mathcal{T} \exp \left\{ -i \int_{-\infty}^{\infty} dx \widehat{\mathcal{H}}_k(x) \right\} \widehat{\Pi}_k$$

$$\widehat{\mathcal{H}}_k(x) := \frac{1}{2} e^{-i\widehat{\varpi}x\sigma_3} \widehat{V}_k(x) \widehat{\varpi}^{-1} \mathcal{K} e^{i\widehat{\varpi}x\sigma_3} = \widehat{\Pi}_k \widehat{\mathcal{H}}(x) \widehat{\Pi}_k$$

Transfer matrix for \mathcal{V}_k :

Recall: $\widehat{\mathbf{M}} : \mathcal{F}_k^2 \rightarrow \mathcal{F}_k^2$,

$$\widehat{\mathbf{M}} := \widehat{\Pi}_k \mathcal{T} \exp \left\{ -i \int_{-\infty}^{\infty} dx \widehat{\mathcal{H}}(x) \right\} \widehat{\Pi}_k$$

$$\widehat{\mathcal{H}}(x) := \frac{1}{2} e^{-i\widehat{\varpi}x\sigma_3} v(x, \widehat{y}) \widehat{\varpi}^{-1} \mathcal{K} e^{i\widehat{\varpi}x\sigma_3}$$

$$\begin{aligned} \text{PWA} : v \longrightarrow \mathcal{V}_k &\quad \Leftrightarrow \quad v(x, \widehat{y}) \longrightarrow \widehat{V}_k(x) := \widehat{\Pi}_k v(x, \widehat{y}) \widehat{\Pi}_k \\ &\quad \Rightarrow \quad \widehat{\Pi}_k \widehat{V}_k(x) \widehat{\Pi}_k = \widehat{V}_k(x) \end{aligned}$$

$$\widehat{\mathbf{M}} \longrightarrow \widehat{\mathbf{M}}_k := \widehat{\Pi}_k \mathcal{T} \exp \left\{ -i \int_{-\infty}^{\infty} dx \widehat{\mathcal{H}}_k(x) \right\} \widehat{\Pi}_k$$

$$\begin{aligned} \widehat{\mathcal{H}}_k(x) &:= \frac{1}{2} e^{-i\widehat{\varpi}x\sigma_3} \widehat{V}_k(x) \widehat{\varpi}^{-1} \mathcal{K} e^{i\widehat{\varpi}x\sigma_3} = \widehat{\Pi}_k \widehat{\mathcal{H}}(x) \widehat{\Pi}_k \\ &= \frac{1}{2} e^{-i\widehat{\varpi}x\sigma_3} \widehat{\Pi}_k \widehat{V}_k(x) \widehat{\Pi}_k \widehat{\varpi}^{-1} \mathcal{K} e^{i\widehat{\varpi}x\sigma_3} \end{aligned}$$

Transfer matrix for \mathcal{V}_k :

Recall: $\widehat{\mathbf{M}} : \mathcal{F}_k^2 \rightarrow \mathcal{F}_k^2$,

$$\widehat{\mathbf{M}} := \widehat{\Pi}_k \mathcal{T} \exp \left\{ -i \int_{-\infty}^{\infty} dx \widehat{\mathcal{H}}(x) \right\} \widehat{\Pi}_k$$

$$\widehat{\mathcal{H}}(x) := \frac{1}{2} e^{-i\widehat{\varpi}x\sigma_3} v(x, \widehat{y}) \widehat{\varpi}^{-1} \mathcal{K} e^{i\widehat{\varpi}x\sigma_3}$$

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$$\widehat{\mathbf{M}} \longrightarrow \widehat{\mathbf{M}}_k := \widehat{\Pi}_k \mathcal{T} \exp \left\{ -i \int_{-\infty}^{\infty} dx \widehat{\mathcal{H}}_k(x) \right\} \widehat{\Pi}_k$$

$$\begin{aligned} \widehat{\mathcal{H}}_k(x) &:= \frac{1}{2} e^{-i\widehat{\varpi}x\sigma_3} \widehat{V}_k(x) \widehat{\varpi}^{-1} \mathcal{K} e^{i\widehat{\varpi}x\sigma_3} = \widehat{\Pi}_k \widehat{\mathcal{H}}(x) \widehat{\Pi}_k \\ &= \frac{1}{2} e^{-i\widehat{\varpi}x\sigma_3} \widehat{\Pi}_k \widehat{V}_k(x) \widehat{\Pi}_k \widehat{\varpi}^{-1} \mathcal{K} e^{i\widehat{\varpi}x\sigma_3} \\ &= \widehat{\Pi}_k \widehat{\mathcal{H}}_k(x) \widehat{\Pi}_k \end{aligned}$$

Transfer matrix for \mathcal{V}_k :

Recall: $\widehat{\mathbf{M}} : \mathcal{F}_k^2 \rightarrow \mathcal{F}_k^2$,

$$\widehat{\mathbf{M}} := \widehat{\Pi}_k \mathcal{T} \exp \left\{ -i \int_{-\infty}^{\infty} dx \widehat{\mathcal{H}}(x) \right\} \widehat{\Pi}_k$$

$$\widehat{\mathcal{H}}(x) := \frac{1}{2} e^{-i\widehat{\varpi}x\sigma_3} v(x, \widehat{y}) \widehat{\varpi}^{-1} \mathcal{K} e^{i\widehat{\varpi}x\sigma_3}$$

$$\begin{aligned} \text{PWA} : v \longrightarrow \mathcal{V}_k &\quad \Leftrightarrow \quad v(x, \widehat{y}) \longrightarrow \widehat{V}_k(x) := \widehat{\Pi}_k v(x, \widehat{y}) \widehat{\Pi}_k \\ &\quad \Rightarrow \quad \widehat{\Pi}_k \widehat{V}_k(x) \widehat{\Pi}_k = \widehat{V}_k(x) \end{aligned}$$

$$\widehat{\mathbf{M}} \longrightarrow \widehat{\mathbf{M}}_k := \widehat{\Pi}_k \mathcal{T} \exp \left\{ -i \int_{-\infty}^{\infty} dx \widehat{\mathcal{H}}_k(x) \right\} \widehat{\Pi}_k$$

$$\begin{aligned} \widehat{\mathcal{H}}_k(x) &:= \frac{1}{2} e^{-i\widehat{\varpi}x\sigma_3} \widehat{V}_k(x) \widehat{\varpi}^{-1} \mathcal{K} e^{i\widehat{\varpi}x\sigma_3} = \widehat{\Pi}_k \widehat{\mathcal{H}}(x) \widehat{\Pi}_k \\ &= \frac{1}{2} e^{-i\widehat{\varpi}x\sigma_3} \widehat{\Pi}_k \widehat{V}_k(x) \widehat{\Pi}_k \widehat{\varpi}^{-1} \mathcal{K} e^{i\widehat{\varpi}x\sigma_3} \\ &= \widehat{\Pi}_k \widehat{\mathcal{H}}_k(x) \widehat{\Pi}_k \end{aligned}$$

$$\Rightarrow \quad \widehat{\mathbf{M}}_k := \mathcal{T} \exp \left\{ -i \int_{-\infty}^{\infty} dx \widehat{\mathcal{H}}_k(x) \right\}$$

$$\begin{aligned}
\widehat{\mathbf{M}} &:= \widehat{\Pi}_k \mathcal{T} \exp \left\{ -i \int_{-\infty}^{\infty} dx \widehat{\mathcal{H}}(x) \right\} \widehat{\Pi}_k \\
&= \widehat{\mathbf{I}} - i \int_{-\infty}^{\infty} dx_1 \widehat{\Pi}_k \widehat{\mathcal{H}}(x_1) \widehat{\Pi}_k \\
&\quad - \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{x_2} dx_1 \widehat{\Pi}_k \widehat{\mathcal{H}}(x_2) \widehat{\mathcal{H}}(x_1) \widehat{\Pi}_k + \cdots
\end{aligned}$$

$$\widehat{\mathcal{H}}(x) := \frac{1}{2} e^{-i \widehat{\varpi} x \sigma_3} v(x, \widehat{y}) \widehat{\varpi}^{-1} \kappa e^{i \widehat{\varpi} x \sigma_3}$$

$$\widehat{\Pi}_k \widehat{\mathcal{H}}(x_1) \widehat{\Pi}_k = \widehat{\mathcal{H}}_k(x_1)$$

$$\begin{aligned}
\widehat{\mathbf{M}} &:= \widehat{\Pi}_k \mathcal{T} \exp \left\{ -i \int_{-\infty}^{\infty} dx \widehat{\mathcal{H}}(x) \right\} \widehat{\Pi}_k \\
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&\quad - \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{x_2} dx_1 \widehat{\Pi}_k \widehat{\mathcal{H}}(x_2) \widehat{\mathcal{H}}(x_1) \widehat{\Pi}_k + \cdots
\end{aligned}$$

$$\widehat{\mathcal{H}}(x) := \frac{1}{2} e^{-i \widehat{\varpi} x \sigma_3} v(x, \widehat{y}) \widehat{\varpi}^{-1} \mathcal{K} e^{i \widehat{\varpi} x \sigma_3}$$

$$\widehat{\Pi}_k \widehat{\mathcal{H}}(x_1) \widehat{\Pi}_k = \widehat{\mathcal{H}}_k(x_1)$$

If $\widehat{\mathcal{H}}(x_2) \widehat{\mathcal{H}}(x_1) \approx \widehat{0}$ for all $x_1, x_2 \in \mathbb{R}$, $\widehat{\mathbf{M}} \approx \widehat{\mathbf{M}}_k$.

\Rightarrow **PWA** is valid for weak potentials.

$$\begin{aligned}
\widehat{\mathbf{M}} &:= \widehat{\Pi}_k \mathcal{T} \exp \left\{ -i \int_{-\infty}^{\infty} dx \widehat{\mathcal{H}}(x) \right\} \widehat{\Pi}_k \\
&= \widehat{\mathbf{I}} - i \int_{-\infty}^{\infty} dx_1 \widehat{\Pi}_k \widehat{\mathcal{H}}(x_1) \widehat{\Pi}_k \\
&\quad - \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{x_2} dx_1 \widehat{\Pi}_k \widehat{\mathcal{H}}(x_2) \widehat{\mathcal{H}}(x_1) \widehat{\Pi}_k + \cdots
\end{aligned}$$

$$\widehat{\mathcal{H}}(x) := \frac{1}{2} e^{-i \widehat{\varpi} x \sigma_3} v(x, \widehat{y}) \widehat{\varpi}^{-1} \mathcal{K} e^{i \widehat{\varpi} x \sigma_3}$$

$$\widehat{\Pi}_k \widehat{\mathcal{H}}(x_1) \widehat{\Pi}_k = \widehat{\mathcal{H}}_k(x_1)$$

If $\widehat{\mathcal{H}}(x_2) \widehat{\mathcal{H}}(x_1) \approx \widehat{0}$ for all $x_1, x_2 \in \mathbb{R}$, $\widehat{\mathbf{M}} \approx \widehat{\mathbf{M}}_k$.

\Rightarrow **PWA** is valid for weak potentials.

For $v(x, y) = \delta(x)g(y)$, $\widehat{\mathcal{H}}(x_2) \widehat{\mathcal{H}}(x_1) = \widehat{0}$ & $\widehat{\mathbf{M}} = \widehat{\mathbf{M}}_k$

\Rightarrow **PWA** is exact.

[arXiv: 2204.05153]

A class of potentials for which PWA is exact:

Theorem: Let $v : \mathbb{R}^2 \rightarrow \mathbb{C}$ be a potential such that

$$\tilde{v}(x, p) = 0 \quad \text{for } p \leq 0, \quad \text{or} \quad \tilde{v}(x, p) = 0 \quad \text{for } p \geq 0.$$

Then PWA is exact for v .

A class of potentials for which **PWA** is exact:

Theorem: Let $v : \mathbb{R}^2 \rightarrow \mathbb{C}$ be a potential such that $\tilde{v}(x, p) = 0$ for $p \leq 0$, or $\tilde{v}(x, p) = 0$ for $p \geq 0$. Then **PWA** is exact for v .

Example: Let $g : \mathbb{R} \rightarrow \mathbb{C}$ be a function such that

$$x g(x) \rightarrow 0 \quad \text{for } x \rightarrow \pm\infty,$$

and

$$v(x, y) = \frac{g(x)}{(\alpha - iy)^{\ell+1}}, \quad \alpha \in \mathbb{R}^+, \ell \in \mathbb{Z}^+.$$

Then $\tilde{v}(x, p) = 0$ for $p \leq 0$, and **PWA** is exact for v .

Existence of the transfer matrix

$$\widehat{\mathbf{M}} := \widehat{\mathcal{U}}(\infty, -\infty) := \lim_{x_{\pm} \rightarrow \pm\infty} \widehat{\mathcal{U}}(x_+, x_-)$$

$$\widehat{\mathcal{U}}(x, x_0) := \widehat{\mathbf{I}} - i \int_{x_0}^x dx_1 \widehat{\Pi}_k \widehat{\mathcal{H}}(x_1) \widehat{\Pi}_k - \int_{x_0}^x dx_2 \int_{x_0}^{x_2} dx_1 \widehat{\Pi}_k \widehat{\mathcal{H}}(x_2) \widehat{\mathcal{H}}(x_1) \widehat{\Pi}_k + \cdots$$

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We could answer these questions for $\widehat{\mathcal{H}}_k(x)$ and prove the existence of $\widehat{\mathbf{M}}_k$.

[arXiv: 2207.10054]

Definition: Let $v : \mathbb{R}^2 \rightarrow \mathbb{C}$ be a bounded integrable function and $s, k \in \mathbb{R}^+$. We say that $v \in \mathcal{C}_{s,k}$ if

1) $\exists \alpha, \beta, \sigma \in \mathbb{R}^+$ such that $\sigma > s$ and for all $(x, y) \in \mathbb{R}^2$,

$$|v(x, y)| \leq \frac{\beta}{(1 + |x|)^\sigma} \quad \text{for } |x| \geq \alpha.$$

2) The function $\tilde{v}_k : \mathbb{R} \rightarrow L^\infty(-2k, 2k)$ defined by

$$(\tilde{v}_k(x))(p) := \tilde{v}(x, p), \quad x \in \mathbb{R}, \quad p \in (-2k, 2k),$$

is piecewise continuous.

Theorem: Let $v \in \mathcal{C}_{3,k}$. Then the following hold.

1) For all $x \in \mathbb{R}$, $\hat{\mathcal{H}}_k(x)$ defines an unbounded non-self-adjoint linear operator acting in

$$\mathcal{H} := L^2(-k, k)^2 = \left\{ \begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix} \mid \phi_{\pm} \in L^2(-k, k) \right\}$$

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and having the domain:

$$\mathcal{D} = \left\{ \begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix} \in \mathcal{H} \mid \phi_+ + \phi_- \in \mathcal{R} \right\},$$

where $\mathcal{R} := \left\{ \sqrt{k^2 - \hat{p}^2} \phi \mid \phi \in L^2(-k, k) \right\},$

$$(\sqrt{k^2 - \hat{p}^2} \phi)(p) := \sqrt{k^2 - p^2} \phi(p) \quad \text{for } p \in (-k, k)$$

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2) The Dyson series for $\hat{\mathcal{U}}_k(x, x_0)$ converges strongly on \mathcal{D} .

3) $\lim_{x_{\pm} \rightarrow \pm\infty} \hat{\mathcal{U}}(x_+, x_-)$ exist as strong limits on \mathcal{D} .

[arXiv: 2207.10054]

Conclusions

- Propagating-wave approximation (PWA)
- Scattering by nonlocal potential achieving PWA
- Existence of the transfer matrix for these potentials
- Numerical implementations: $L^2(-k, k)$ & Hilbert-Schmidt (compact) operators.

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Thank you for your attention.