SOLITON GENERATION IN OPTICAL FIBER NETWORKS

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Prague, September 7, 2022

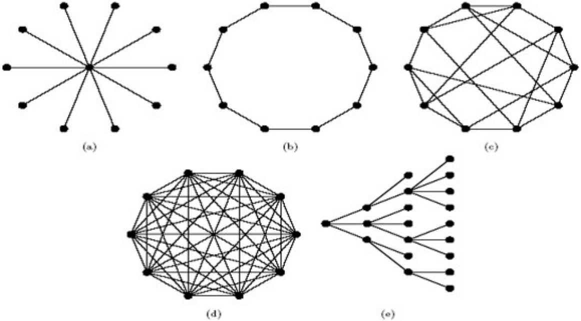
Outline

Motivation

Why networks, why solitons in networks, why PT-symmetry?

- Most of the optical, electronic and opto-eletronic devices have branched structure;
- Energy, signal and information transfer through the networks is a key process in modern technologies;
- Controlling, manipulating particle/wave transport in networks by tuning network architecture (topology) and nodes transfer properties provides effective tool for functional optimization of the devices and materials;
- Solitons are the signal, energy and charge carriers in many optical, electronic, optoelectronic, condensed matter and even biological structures which often have branched architectures;
- PT-symmetric nonlinear structures are the new forms of materials having challenging practical applications in newly emerging technologies.

Graphs and their topology



(a) star graph ($B = 10$, $V = 11$),
(b) ring graph ($B = 10$, $V = 10$),
(c) v-regular graph with $v = 4$ ($B = 20$, $V = 10$),
(d) complete (or well-connected) graph ($B = 45$, $V = 10$),
(e) tree graph ($B = 19$, $V = 20$).

Graphs and their topology

The topology of the graph, that is, the way the vertices and bonds are connected is given in terms of the VxV connectivity matrix $C_{i,j}$ (sometimes referred to as the adjacency matrix) which is defined as:

$$C_{i,j} = C_{j,i} = \left\{ \begin{array}{cc} 1 & \text{if } \; i,j \; \text{ are connected} \\ 0 & \text{otherwise} \end{array} \right\}, \quad i,j = 1, ..., V$$

Metric graphs

A graph with the bonds which can be assigned length,

$$0 < l_b :< D$$

is called metric graph.

Evolution equations on graphs

- Metric graphs are the effective tool for modeling networks and branched domains;
- They greatly simplify modeling wave dynamics in branched structures;
- The problem of tunable wave transport;
- Dimensional reduction;
- Opening closing the nodes, directed motion.

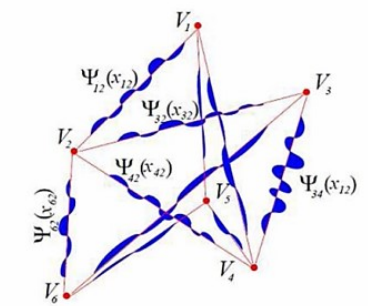
Nonlinear evolution equations on networks

1. Nonlinear Schrödinger equation;
2. Sine-Gordon equation;
3. Korteweg-de Vries equation;
4. Nonlinear Dirac equation;
5. Burgers equation;
6. And other nonlinear wave equations.

Evolution equation on graphs

$$i\frac{\partial\vec{\psi}}{\partial t} = H\vec{\psi}$$

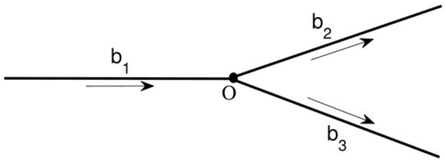
where H is the Schrödinger, Dirac, or other differential operator. Wave function $\vec{\psi}$ is a $B-$component vector $\vec{\psi} = (\psi_{b_1}(x_{b_1}), \psi_{b_2}(x_{b_2}), \psi_{b_3}(x_{b_3}), ..., \psi_{b_B}(x_{b_B}))^T$.



Nonlinear evolution equations on metric graphs: Our approach

1. Write a given PDE on a graph;
2. Find/define its solution on a (infinite) line;
3. Define conserving quantities for the PDE;
4. Derive vertex boundary conditions from the conservation laws;
5. Require fulfilling vertex boundary conditions by the solution on a line and derive constraints ensuring such fulfilling;
6. Numerical solution of the problem, when the above constraints are not fulfilled;
7. Simulation of the soliton dynamics on a graph.

Nonlinear Schrödinger equation on graphs



Defining of the bonds:

$$b_1 \sim (-\infty, 0], \quad b_{2,3} \sim [0, +\infty)]$$

NLS equation on the each bond of the star graph[1]:

$$i\frac{\partial \psi_k}{\partial t} + \frac{\partial^2 \psi_k}{\partial x^2} + \beta_k |\psi_k|^2 \psi_k = 0, \quad x_k \in b_k, \quad k = 1, 2, 3$$

[1] Z. Sobirov, D. Matrasulov, K. Sabirov, S. Sawada, and K. Nakamura, Phys. Rev. E 81, 066602 (2010).

Soliton solution of NLSE on a line

NLS equation on a line:

$$i\frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + 2|\psi|^2\psi = 0$$

soliton solution:

$$\psi(x,t) = \eta \mathrm{sech}[\eta(x + x_0 - ct)]e^{-i[2cx-(c^2-4\eta^2)t]/4}$$

$\eta-$is the amplitude, $x_0-$is the center and $c-$is the velocity of the soliton.

Conservative quantities

The norm:

$$N = \sum_{b_k} \int_{b_k} |\psi_k|^2 dx.$$

Energy:

$$E = \sum_{b_k} \int_{b_k} \left(\left| \frac{\partial \psi_k}{\partial x} \right|^2 - \frac{\beta_k}{2} |\psi_k|^4 \right) dx$$

Conservative quantities

Norm – const:

$$\mathrm{Im}\left[\psi_1^*\frac{\partial\psi_1}{\partial x}\right]\Big|_{x=0}=\mathrm{Im}\left[\psi_2^*\frac{\partial\psi_2}{\partial x}\right]\Big|_{x=0}+\mathrm{Im}\left[\psi_3^*\frac{\partial\psi_3}{\partial x}\right]\Big|_{x=0}$$

Energy – const:

$$\mathrm{Re}\left[\frac{\partial\psi_1^*}{\partial x}\frac{\partial\psi_1}{\partial t}\right]\Big|_{x=0}=\mathrm{Re}\left[\frac{\partial\psi_2^*}{\partial x}\frac{\partial\psi_2}{\partial t}\right]\Big|_{x=0}+\mathrm{Re}\left[\frac{\partial\psi_3^*}{\partial x}\frac{\partial\psi_3}{\partial t}\right]\Big|_{x=0}.$$

Vertex boundary conditions

- The first choice:

$$\alpha_1\psi_1|_{x=0} = \alpha_2\psi_2|_{x=0} = \alpha_3\psi_3|_{x=0}$$
$$\frac{1}{\alpha_1}\frac{\partial\psi_1}{\partial x}\bigg|_{x=0} = \frac{1}{\alpha_2}\frac{\partial\psi_2}{\partial x}\bigg|_{x=0} + \frac{1}{\alpha_3}\frac{\partial\psi_3}{\partial x}\bigg|_{x=0}$$

- The second choice:

$$\frac{1}{\alpha_1}\psi_1|_{x=0} = \frac{1}{\alpha_2}\psi_2|_{x=0} + \frac{1}{\alpha_3}\psi_3|_{x=0}$$
$$\alpha_1\frac{\partial\psi_1}{\partial x}\bigg|_{x=0} = \alpha_2\frac{\partial\psi_2}{\partial x}\bigg|_{x=0} = \alpha_3\frac{\partial\psi_3}{\partial x}\bigg|_{x=0}$$

[1] Z. Sobirov, D. Matrasulov, K. Sabirov, S. Sawada, and K. Nakamura, Phys. Rev. E 81, 066602 (2010).

Soliton solution and the constraint

In particular at $\alpha_k = \sqrt{\beta_k}$ the wave function can be found in the form

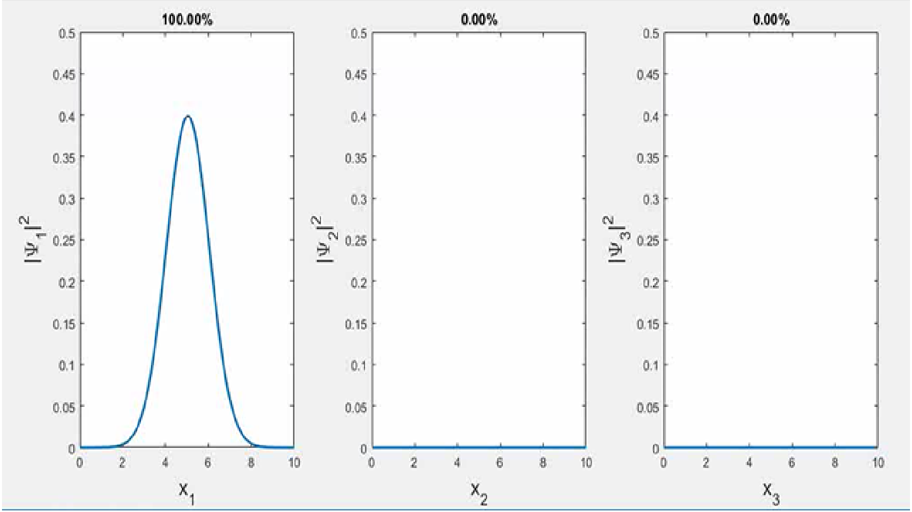
$$\psi_k(x,t) = i\sqrt{\frac{2}{\beta_k}}\psi(x,t)$$

Where $q(x,t)$ is the solution of NLSE (on a line).
Boundary conditions are satisfied if

$$\frac{1}{\beta_1} = \frac{1}{\beta_2} + \frac{1}{\beta_3}$$

[1] Z. Sobirov, D. Matrasulov, K. Sabirov, S. Sawada, and K. Nakamura, Phys. Rev. E 81, 066602 (2010).

Soliton dynamics



Soliton generation on the line

$$i\frac{\partial \psi}{\partial t}+\frac{1}{2}\frac{\partial^2 \psi}{\partial x^2}+|\psi|^2\psi=0$$

The number of generated solitons[2]

$$N=\left\langle \frac{1}{2}+\frac{F}{\pi}\right\rangle$$

where $F=\int_{-\infty}^{+\infty}|\psi(x,0)|dx$.

[2] N.-C. Panoiu, I. V. Mel'nikov, D. Mihalache, C. Etrich, and F. Lederer, Phys. Rev. E **60**, 4868 (1999).

Example for soliton generation

The initial pulse profile[3]

$$\psi(x,0) = -iq(x) \qquad q(x) = \begin{cases} 0 & \text{for} \quad |x| > a/2 \\ b & \text{for} \quad |x| \leq a/2 \end{cases}$$

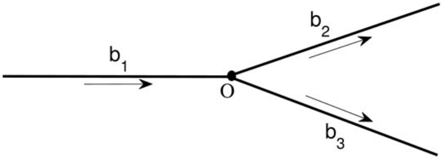
$$N = \left\langle \frac{1}{2} + \frac{ab}{\pi} \right\rangle$$

[3] J. Burzlaff, J. Phys. A: Math. Gen. **21** (1988), 561-566.

Nonlinear Schrödinger equation on a star graph

$$i\frac{\partial \psi_j}{\partial t} + \frac{\partial^2 \psi_j}{\partial x^2} + \beta_j |\psi_j|^2 \psi_j = 0$$

where $j = 1, 2, 3$.



Zakharov–Shabat problem on the star graph

$$\frac{\partial v_j^{(1)}}{\partial x} = i\lambda v_j^{(1)} + i\psi_j(x,0)v_j^{(2)},$$
$$\frac{\partial v_j^{(2)}}{\partial x} = -i\lambda v_j^{(2)} + i\psi_j^*(x,0)v_j^{(1)},$$

Let us consider the special family of the initial potentials

$$\psi_j(x,0) = \Psi_j(x,0)e^{i\delta_j},$$

and eigenfunction transformations

$$v_j^{(1)} \to V_j^{(1)}e^{i\gamma_j},\ v_j^{(2)} \to V_j^{(2)}e^{i(\gamma_j-\delta_j)}$$

Zakharov–Shabat problem on the star graph

$$\frac{\partial V_j^{(1)}}{\partial x} = i\lambda V_j^{(1)} + i\Psi_j(x,0)V_j^{(2)},$$
$$\frac{\partial V_j^{(2)}}{\partial x} = -i\lambda V_j^{(2)} + i\Psi_j(x,0)V_j^{(1)}.$$

We will define the number of the zeros of the Jost coefficients $a_j(\lambda)$ at $\lambda = 0$.

$$\frac{\partial V_j^{(1)}}{\partial x} = i\Psi_j(x,0)V_j^{(2)},$$
$$\frac{\partial V_j^{(2)}}{\partial x} = i\Psi_j(x,0)V_j^{(1)}.$$

Explicit form of the eigenfunctions

$$V_1^{(1)}(x,0)=\exp\left(-iS_1(x)\right)\left(C_1^{(1)}\int\limits_{-\infty}^{x}\Psi_1(x',0)\exp\left(2iS_1(x')\right)dx'+C_1^{(2)}\right),$$

$$V_1^{(2)}(x,0)=-iC_1^{(1)}\exp\left(iS_1(x)\right)-V_1^{(1)},$$

$$V_{2,3}^{(1)}(x,0)=\exp\left(-iS_{2,3}(x)\right)\left(C_{2,3}^{(1)}\int\limits_{0}^{x}\Psi_{2,3}(x',0)\exp\left(2iS_{2,3}(x')\right)dx'+C_{2,3}^{(2)}\right),$$

$$V_{2,3}^{(2)}(x,0)=-iC_{2,3}^{(1)}\exp\left(iS_{2,3}(x)\right)-V_{2,3}^{(1)},$$

where $S_1(x)=\int\limits_{-\infty}^{x}\Psi_1(x',0)dx'$, and $S_{2,3}(x)=\int\limits_{0}^{x}\Psi_{2,3}(x',0)dx'$.

Deriving the soliton number

If one chooses $V_1^{(1)}(x,0) \to 0$ for $x \to -\infty$ and $V_{2,3}^{(1)}(x,0) \to 0$ for $x \to -0$, then $C_j^{(2)} = 0$, and we have

$$
\begin{aligned}
a_j(0) &= \lim_{x \to \pm 0} V_j^{(2)}(x,0) \\
&= -iC_j^{(1)} \left(\exp(iS_{0j}) - i \exp(-iS_{0j}) \int_{b_j} \Psi_j(x,0) \exp(2iS_j(x)) dx \right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = -iC_j^{(1)} \cos S_{0j},
\end{aligned}
$$

where $S_{0j} = \int_{b_j} \Psi_j(x,0) dx$.

Deriving the soliton number

The soliton number on the star graph[4]

$$N = \left\langle \frac{3}{2} + \frac{S_{01} + S_{02} + S_{03}}{\pi} \right\rangle .$$

Noting that for the initial pulses for any x and with $\Psi_j(x,0) > 0$

$$S_{0j} \equiv \int\limits_{b_j} \Psi_j(x,0)dx = \int\limits_{b_j} |\psi_j(x,0)|dx = F_j.$$

we have

$$N = \left\langle \frac{3}{2} + \frac{F_1 + F_2 + F_3}{\pi} \right\rangle .$$

[4] K.K Sabirov, M.E Akramov, R.Sh Otajonov, D.U Matrasulov, Chaos, Solitons & Fractals **133**, 109636 (2020).

The initial pulse profile on the star graph

$$\psi_j(x,0) = -i\sqrt{\frac{2}{\beta_j}}q_j(x)$$

$$q_1(x) = \begin{cases} 0, & x < -\frac{1}{2}a \\ b, & -\frac{1}{2}a \le x \le 0 \end{cases}$$

$$q_{2,3}(x) = \begin{cases} 0, & x > \frac{1}{2}a \\ b, & 0 \le x \le \frac{1}{2}a \end{cases}$$

The soliton number for the above initial condition

$$N = \left\langle \frac{3}{2} + \frac{F}{\pi} \right\rangle, \quad F = \sum_{j=1}^{3} \int_{b_j} |\psi_j(x,0)| dx = \frac{ab}{2}\left[\sqrt{\frac{2}{\beta_1}} + \sqrt{\frac{2}{\beta_2}} + \sqrt{\frac{2}{\beta_3}}\right].$$

The Gaussian input pulse

$$\psi_j(x,0)=\sqrt{\frac{2}{\beta_j}}A\exp\left[-\frac{1}{2}(1-i\alpha)\left(\frac{x}{\sigma}\right)^{2m}\right]$$

The soliton number for Gaussian input pulse

$$N=\left\langle\frac{3}{2}+\frac{F}{\pi}\right\rangle,$$

$$F=\sum_{j=1}^{3}\int_{b_j}|\psi_j(x,0)|dx=\frac{2^{\frac{1}{2m}}A\sigma}{2m}\Gamma\left(\frac{1}{2m}\right)\left[\sqrt{\frac{2}{\beta_1}}+\sqrt{\frac{2}{\beta_2}}+\sqrt{\frac{2}{\beta_3}}\right].$$

Nonlocal nonlinear Schrödinger equation on a line

Nonlocal NLS equation[5]

$$i\frac{\partial}{\partial t}q(x,t) = \frac{\partial^2}{\partial x^2}q(x,t) + 2q^2(x,t)q^*(-x,t).$$

Nonlocal NLS equation can be rewritten as

$$i\frac{\partial}{\partial t}q(x,t) = \frac{\partial^2}{\partial x^2}q(x,t) + V(x,t)q(x,t),$$

where $V = 2q(x,t)q^*(-x,t)$ is the PT-symmetric self-induced potential.

[5] Mark J. Ablowitz and Ziad H. Musslimani, Phys. Rev. Lett. **110**, 064105 (2013).

Soliton generation for nonlocal NLS equation

The soliton number[6]

$$N = \left\langle \frac{1}{2} + \frac{F}{\pi} \right\rangle, \quad \text{where} \quad F = \int\limits_{-\infty}^{+\infty} |q(x,0)| dx$$

Initial pulse profile is given as follows

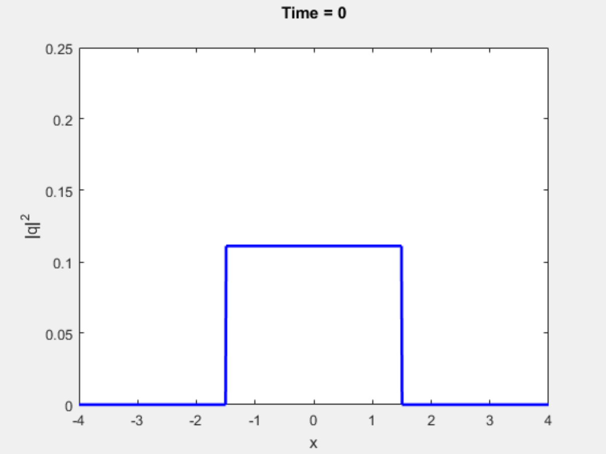
$$q(x,0) = \begin{cases} 0, & \text{for} \quad |x| > \frac{1}{2}a \\ b, & \text{for} \quad |x| \leq \frac{1}{2}a \end{cases} \qquad b > 0.$$

The soliton number

$$N = \left\langle \frac{1}{2} + \frac{ab}{\pi} \right\rangle, \quad F = \int_{-\infty}^{+\infty} |q(x,0)| dx = ab.$$

[6] M. Akramov, K. Sabirov, D. Matrasulov, H. Susanto, S. Usanov, and O. Karpova, Phys. Rev. E **105**, 054205 (2022).

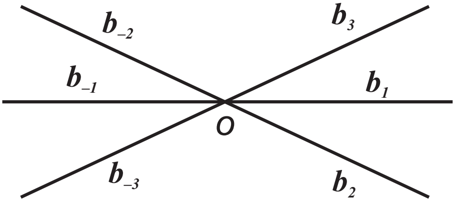
Numerical simulation of soliton generation



Two soliton generation for rectangular initial condition with the parameters: $a = 3$, $b = 2$, i.e., $N = \left\langle \frac{1}{2} + \frac{ab}{\pi} \right\rangle = 2$.

Nonlocal NLS equation on the star graph

$$i\frac{\partial}{\partial t}q_{\pm j}(x,t)=\frac{\partial^2}{\partial x^2}q_{\pm j}(x,t)+\sqrt{\beta_j\beta_{-j}}q_{\pm j}^2(x,t)q_{\mp j}^*(-x,t),$$



[6] M. Akramov, K. Sabirov, D. Matrasulov, H. Susanto, S. Usanov, and O. Karpova, Phys. Rev. E **105**, 054205 (2022).

Vertex boundary conditions

$$\alpha_1 q_1(x,t)|_{x=0} = \alpha_{-1} q_{-1}(x,t)|_{x=0} = \alpha_2 q_2(x,t)|_{x=0}$$
$$= \alpha_{-2} q_{-2}(x,t)|_{x=0} = \alpha_3 q_3(x,t)|_{x=0} = \alpha_{-3} q_{-3}(x,t)|_{x=0},$$

$$\frac{1}{\alpha_1}\frac{\partial}{\partial x} q_1(x,t)\Big|_{x=0} + \frac{1}{\alpha_2}\frac{\partial}{\partial x} q_2(x,t)\Big|_{x=0} + \frac{1}{\alpha_3}\frac{\partial}{\partial x} q_3(x,t)\Big|_{x=0}$$
$$= \frac{1}{\alpha_{-1}}\frac{\partial}{\partial x} q_{-1}(x,t)\Big|_{x=0} + \frac{1}{\alpha_{-2}}\frac{\partial}{\partial x} q_{-2}(x,t)\Big|_{x=0} + \frac{1}{\alpha_{-3}}\frac{\partial}{\partial x} q_{-3}(x,t)\Big|_{x=0}.$$

Sum rule

$$\frac{\alpha_{\pm j}}{\alpha_1} = \sqrt{\frac{\beta_{\pm j}}{\beta_1}}, \qquad \frac{1}{\beta_1} + \frac{1}{\beta_2} + \frac{1}{\beta_3} = \frac{1}{\beta_{-1}} + \frac{1}{\beta_{-2}} + \frac{1}{\beta_{-3}}.$$

Solitons solution on the star graph

Breathing soliton solution:

$$q_{\pm j}(x,t) = -\sqrt{\frac{2}{\beta_{\pm j}}}\frac{2(\eta_1+\bar{\eta}_1)e^{i\bar{\theta}_1}e^{-4i\bar{\eta}_1^2 t}e^{-2\bar{\eta}_1 x}}{1+e^{i(\theta_1+\bar{\theta}_1)}e^{4i(\eta_1^2-\bar{\eta}_1^2)t}e^{-2(\eta_1+\bar{\eta}_1)x}}$$

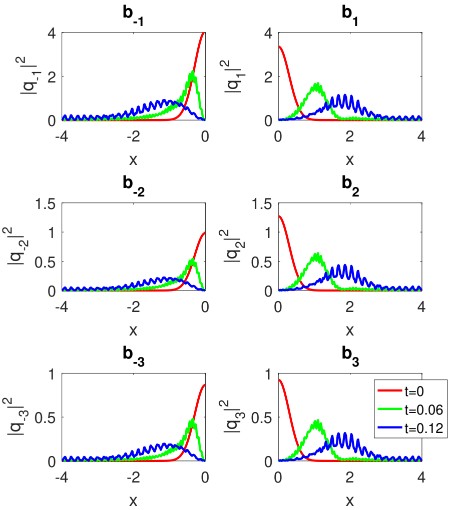
Static soliton solution:

$$q_{\pm j}(x,t) = -\sqrt{\frac{2}{\beta_{\pm j}}}\frac{4\eta e^{i\bar{\varphi}}e^{-4i\eta^2 t}e^{-2\eta x}}{1+e^{i(\varphi+\bar{\varphi})}e^{-4\eta x}}.$$

φ, $\bar{\varphi}$, η are arbitrary complex constants.

Radiation of the input pulse

Gaussian input pulse



Zakharov–Shabat problem on the star graph

$$\frac{\partial v_{\pm j}^{(1)}}{\partial x} = -ikv_{\pm j}^{(1)} + \sqrt{\frac{\beta_{\pm j}}{2}} q_{\pm j}(x,0) v_{\pm j}^{(2)},$$
$$\frac{\partial v_{\pm j}^{(2)}}{\partial x} = ikv_{\pm j}^{(2)} - \sqrt{\frac{\beta_{\mp j}}{2}} q_{\mp j}^{*}(-x,0) v_{\pm j}^{(1)},$$

Note that, the initial condition must be symmetric to $x = 0$ point, $\sqrt{\beta_{\pm j}} q_{\pm j}(x,0) = \sqrt{\beta_{\mp j}} q_{\mp j}^{*}(-x,0)$.

The number of generated solitons

$$N_{\pm j} = \left\langle \frac{1}{2} + \frac{F_{\pm j}}{\pi} \right\rangle, \quad N = \sum_{j=1}^{3} (N_{-j} + N_j).$$

where

$$F_{\pm j} = \sqrt{\frac{\beta_{\pm j}}{2}} \int_{b_{\pm j}} |q_{\pm j}(x,0)| dx.$$

[6] M. Akramov, K. Sabirov, D. Matrasulov, H. Susanto, S. Usanov, and O. Karpova, Phys. Rev. E **105**, 054205 (2022).

The initial pulse profile and the number of generated solitons on the star graph

$q_{\pm j}(x,0)=\sqrt{\frac{2}{\beta_{\pm j}}}\psi_{\pm j}(x)$:

$$\psi_{-j}(x)=\begin{cases}0, & \text{for} \quad x<-\frac{1}{2}a\\ b, & \text{for} \quad -\frac{1}{2}a\leq x\leq 0\end{cases},$$

$$\psi_{j}(x)=\begin{cases}0, & \text{for} \quad x>\frac{1}{2}a\\ b, & \text{for} \quad 0\leq x\leq \frac{1}{2}a\end{cases},$$

where $b>0$.
The number of generated solitons can be written as

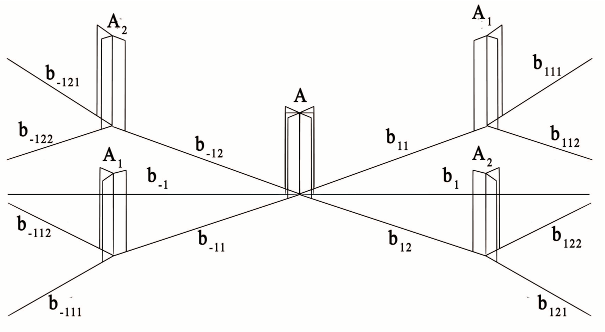
$$N=6\left\langle \frac{1}{2}+\frac{ab}{2\pi}\right\rangle, \quad F_{\pm j}=\sqrt{\frac{\beta_{\pm j}}{2}}\int_{b_{\pm j}}|q_{\pm j}(x,0)|dx=\frac{ab}{2}$$

Extending for a tree graph

Nonlocal NLS equation on tree graph

$$i\frac{\partial}{\partial t}q_{\pm e}(x,t)=\frac{\partial^2}{\partial x^2}q_{\pm e}(x,t)+\sqrt{\beta_e\beta_{-e}}q_{\pm e}^2(x,t)q_{\mp e}^*(-x,t),$$

where $e=\{1,\ 1m,\ 1mn\}$, $n,m=1,2.$



Soliton generation on tree graph

The number of solitons

$$N_e = \left\langle \frac{1}{2} + \frac{F_e}{\pi} \right\rangle, \quad N = \sum_{e\in\Omega} N_e,$$

where $\Omega = \{\pm 1; \pm 1m; \pm 1mn\}$ and

$$F_{\pm 1} = \sqrt{\frac{\beta_{\pm 1}}{2}} \int_{b_{\pm 1}} |q_{\pm 1}(x,0)|dx = \frac{aA}{2},$$

$$F_{\pm 1m} = \sqrt{\frac{\beta_{\pm 1m}}{2}} \int_{b_{\pm 1m}} |q_{\pm 1m}(x,0)|dx = \frac{a}{2}(A + A_m),$$

$$F_{\pm 1mn} = \sqrt{\frac{\beta_{\pm 1mn}}{2}} \int_{b_{\pm 1mn}} |q_{\pm 1mn}(x,0)|dx = \frac{aA_m}{2}.$$

Conslusions

- Nonlinear Schrödinger equation and soliton dynamics on graphs are studied;
- We studied the problem of soliton generation in optical fiber networks using a model based ordinary and nonlocal NLS equation on metric graphs;
- Initial value (Cauchy) problem for ordinary and nonlocal NLS equation on metric graphs is solved for different graph topologies, such as star and tree;
- Analytic expression of number of generated solitons is proved by numerically;
- It was shown that the number of generated solitons depends not on the shape of the initial condition, but on its initial area;
- It was observed that the soliton profile is radiated when the sum rule is not fulfilled.

References

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