Spectrum of the Laplacian on a domain perturbed by small resonators

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based on joint work with Giuseppe Cardone (University of Naples Federico II)

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Introduction

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for any compact set $K \subset \Omega$ (respectively, $K \subset \mathbb{R}^n \setminus \overline{\Omega}$) we have $K \subset \Omega_\varepsilon$ (respectively, $K \subset \mathbb{R}^n \setminus \Omega_\varepsilon$) if ε is sufficiently small.

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$$\forall k \in \mathbb{N}: \quad \lambda_k(-\Delta^D_{\Omega_\varepsilon}) \to \lambda_k(-\Delta^D_\Omega) \text{ as } \varepsilon \to 0$$

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The situation becomes more subtle for the Laplacian with the **Neumann** or **mixed** boundary conditions.

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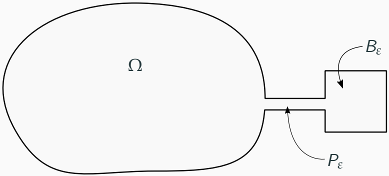

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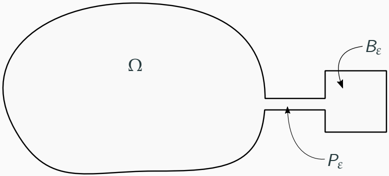

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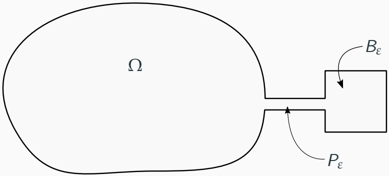

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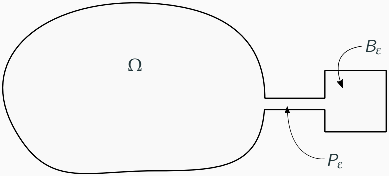

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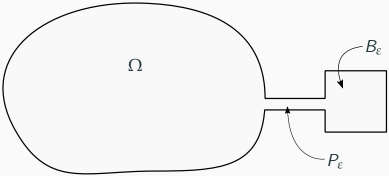

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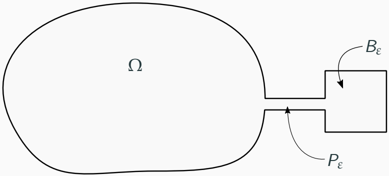

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- $\Delta^N_{\Omega_\varepsilon}$ and Δ^N_Ω are the Neumann Laplacians on Ω_ε and Ω, resp.

It was shown that, if $\alpha > (n+1)/(n-1)$, one has

$$\lim_{\varepsilon\to 0} \lambda_1(-\Delta^N_{\Omega_\varepsilon}) = 0, \quad \lim_{\varepsilon\to 0} \lambda_k(-\Delta^N_{\Omega_\varepsilon}) = \lambda_{k-1}(-\Delta^N_\Omega),\ k \in \mathbb{N} \setminus \{1\}.$$

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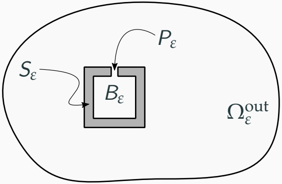

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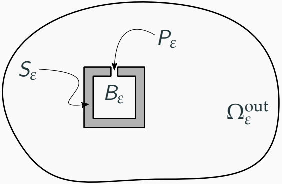

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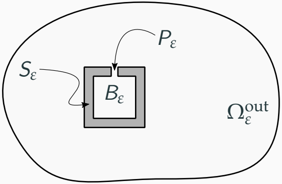

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It was shown that, if B_ε and P_ε are appropriately scaled, one has

$$\lim_{\varepsilon\to 0} \lambda_1(\mathcal{A}_\varepsilon) = \gamma := \lim_{\varepsilon\to 0} \frac{A_\varepsilon}{L_\varepsilon V_\varepsilon}, \quad \lim_{\varepsilon\to 0} \lambda_k(\mathcal{A}_\varepsilon) = \lambda_{k-1}(-\Delta_\Omega^D),\ k \in \mathbb{N}\setminus\{1\},$$

where L_ε and A_ε are the length and the cross-section area of the passage P_ε, respectively, and V_ε is the volume of B_ε.

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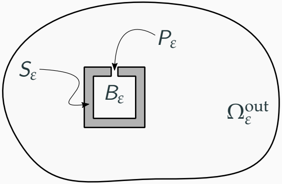

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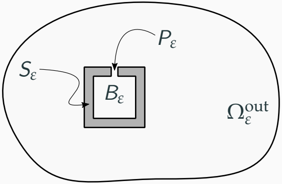

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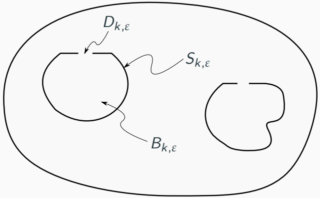
Our goal is to extend and complement the above result to resonators separated from the outer domain by a closed surface with a tiny windows



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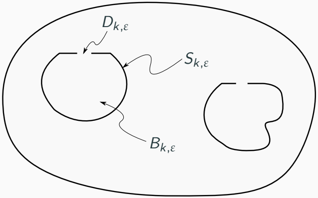

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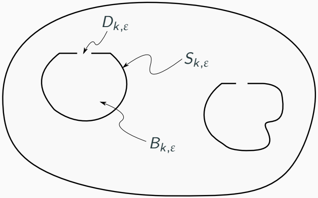

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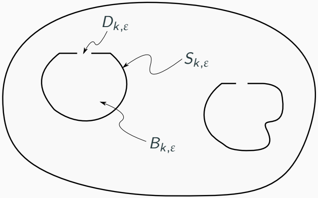

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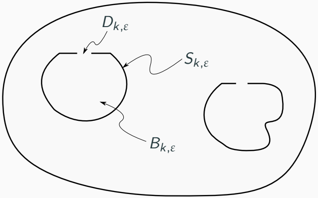

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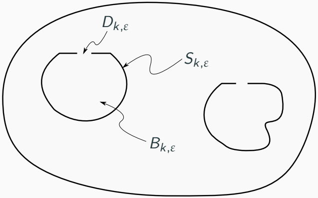

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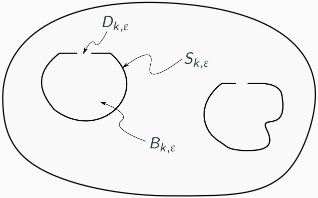

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Main assumption

We assume that the following limits are finite:

$$\gamma_k := \lim_{\varepsilon\to 0} \gamma_{k,\varepsilon},\ \gamma_{k,\varepsilon} = \frac{\mathrm{cap}(D_{k,\varepsilon})}{4|B_{k,\varepsilon}|},\ k \in \{1,\dots,m\} \qquad (*)$$

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Assumption $(*)$ implies

$$d_{k,\varepsilon} = C_\varepsilon \varepsilon^{\frac{n}{n-2}} \text{ if } n\ge 3, \quad |\ln d_{k,\varepsilon}|^{-1} = C_\varepsilon\varepsilon^2 \text{ if } n=2, \quad C_\varepsilon = \mathcal{O}(1).$$

Anticipated limiting operator $\mathcal{A}$

In the Hilbert space $\mathsf{L}^2(\Omega) \oplus \mathbb{C}^m$ we introduce the operator $\mathcal{A}$ via

$$\mathcal{A} = (-\Delta_\Omega^D) \oplus \Gamma,$$

where $\Gamma : \mathbb{C}^m \to \mathbb{C}^m$, $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \dots, \gamma_m)$, and Δ_Ω^D is the Dirichlet Laplacian on Ω.

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Spectrum of $\mathcal{A}$

$$\sigma(\mathcal{A}) = \sigma(-\Delta_\Omega^D) \cup \{\gamma_k,\ k = 1, \ldots, m\}.$$

Hausdorff distance

Hausdorff distance between closed sets $X, Y \subset \mathbb{R}$

$$d_H(X,Y) = \max\Big\{\sup_{x\in X}\inf_{y\in Y}|x-y|;\ \sup_{y\in Y}\inf_{x\in X}|y-x|\Big\}.$$

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Definition

- X_ε *converges from inside to* X ($X_\varepsilon \nearrow X$) if for any $x \in X$ there exists a family $(x_\varepsilon)_{\varepsilon>0}$ with $x_\varepsilon \in X_\varepsilon$ such that $x_\varepsilon \to x$ as $\varepsilon \to 0$;
- X_ε *converges from outside to* X ($X_\varepsilon \searrow X$) if for any $x \in \mathbb{R} \setminus X$ there is $\delta > 0$ such that $X_\varepsilon \cap (x-\delta, x+\delta) = \varnothing$ for small enough ε.
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- X_ε *converges to* X ($X_\varepsilon\to X$) if $X_\varepsilon\nearrow X$ and $X_\varepsilon\searrow X$.

Proposition

Let $(X_\varepsilon)_{\varepsilon>0}$ be a family of closed sets, $X_\varepsilon\subset[0,\infty)$. Let $X\subset[0,\infty)$ be a closed set. Then $\widetilde{d}_H(X_\varepsilon,X)\to 0$ if and only if $X_\varepsilon\to X$.

Main results

Theorem 1

One has $\sigma(\mathcal{A}_\varepsilon) \to \sigma(\mathcal{A})$ as $\varepsilon \to 0$, and the estimate

$$\widetilde{d}_H\left(\sigma(\mathcal{A}_\varepsilon), \sigma(\mathcal{A})\right) \le \begin{cases} C \sum\limits_{k=1}^{m} |\gamma_{k,\varepsilon} - \gamma_k| + C\varepsilon, & n \ge 3, \\ C \sum\limits_{k=1}^{m} |\gamma_{k,\varepsilon} - \gamma_k| + C\varepsilon |\ln \varepsilon|^{3/2}, & n = 2. \end{cases}$$

Main results

Theorem 2

One has $\sigma_{\mathrm{ess}}(\mathcal{A}_\varepsilon) = \sigma_{\mathrm{ess}}(-\Delta_\Omega^D) = \sigma_{\mathrm{ess}}(\mathcal{A})$ and

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The multiplicity is preserved: if $\lambda \in \sigma_{\mathrm{disc}}(\mathcal{A})$ is of multiplicity μ and $[\lambda - L, \lambda + L] \cap \sigma(\mathcal{A}) = \{\lambda\}$ with $L > 0$, then for sufficiently small ε the spectrum of $\mathcal{A}_\varepsilon$ in $[\lambda - L, \lambda + L]$ is purely discrete and the total multiplicity of the eigenvalues of $\mathcal{A}_\varepsilon$ in $[\lambda - L, \lambda + L]$ equals μ.

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If, in addition, $\mu = 1$ (i.e. the eigenvalue λ is simple), and $\psi = (\psi_0, \psi_1, \ldots, \psi_m)$ with $\psi_0 \in \mathsf{L}^2(\Omega)$ and $(\psi_k)_{k=1}^m \in \mathbb{C}^m$ is the corresponding normalized in $\mathsf{L}^2(\Omega) \oplus \mathbb{C}^m$ eigenfunction, then there is a sequence of normalized in $\mathsf{L}^2(\Omega_\varepsilon)$ eigenfunctions ψ_ε of $\mathcal{A}_\varepsilon$ such that

$$\|\psi_\varepsilon - \psi_0\|^2_{\mathsf{L}^2(\Omega_\varepsilon \setminus \overline{\cup_{k=1}^m B_{k,\varepsilon}})} + \sum_{k=1}^m \|\psi_\varepsilon - |B_{k,\varepsilon}|^{-1/2}\psi_k\|^2_{\mathsf{L}^2(B_{k,\varepsilon})} \to 0.$$

Remark

If Ω is a bounded domain, the spectra of $\mathcal{A}_\varepsilon$ and $\mathcal{A}$ are purely discrete; hence, due to Theorem 1, additionally to $\sigma_{\rm disc}(\mathcal{A}_\varepsilon) \nearrow \sigma_{\rm disc}(\mathcal{A})$ we also have $\sigma_{\rm disc}(\mathcal{A}_\varepsilon) \searrow \sigma_{\rm disc}(\mathcal{A})$.

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However, if Ω is unbounded, the latter property does not necessary hold true: it may happen that there exists a sequence $(\lambda_{\varepsilon_k})_{k\in\mathbb{N}}$ with $\lambda_{\varepsilon_k} \in \sigma_{\mathrm{disc}}(\mathcal{A}_{\varepsilon_k})$ converging to $\lambda_0 \in \sigma_{\mathrm{ess}}(\mathcal{A})$.

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Corollary

Let Ω be a bounded domain (consequently, the spectra of $\mathcal{A}_\varepsilon$ and $\mathcal{A}$ are purely discrete). Then $\forall k \in \mathbb{N}$ one has

$$\lambda_k(\mathcal{A}_\varepsilon) \to \lambda_k(\mathcal{A}) \text{ as } \varepsilon \to 0.$$

Application: waveguide with predefined eigenvalues

We are inspired by celebrated papers

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where a Riemannian metric g on a given compact manifold M is constructed such that the first m eigenvalues of the Laplace-Beltrami operator on (M, g) coincide with prescribed numbers, and

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Behrndt, J.; K.A. *Math. Nachr.* **295**, 1063–1095 (2022)

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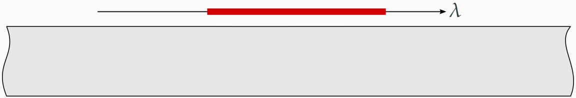

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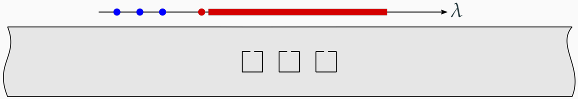

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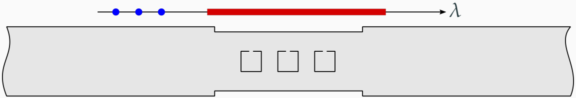

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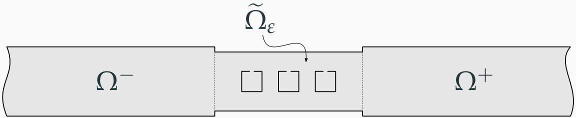

- $\widetilde{\Omega} = \Omega' \times (-L, L)$, where $L > 0$ and Ω' is a bounded Lipschitz domain in $\mathbb{R}^{n-1}$.
- $\widetilde{\Omega}_\varepsilon = \widetilde{\Omega} \setminus (\cup_{k=1}^m S_{k,\varepsilon})$, where

$$S_{k,\varepsilon} = \partial B_{k,\varepsilon} \setminus D_{k,\varepsilon}, \quad B_{k,\varepsilon} \cong \varepsilon B, \quad D_{k,\varepsilon} \cong d_{k,\varepsilon} D,$$

$$d_{k,\varepsilon} = d_k \varepsilon^{\frac{n}{n-2}} \text{ if } n \geq 3, \quad \text{and} \quad d_{k,\varepsilon} = \exp(-1/(d_k \varepsilon^2)) \text{ if } n = 2.$$

- $\Omega^+ = \Omega'' \times (L, \infty)$, $\Omega^- = \Omega'' \times (-\infty, -L)$, where Ω'' is a bounded Lipschitz domain in $\mathbb{R}^{n-1}$ such that $\overline{\Omega'} \subset \Omega''$.
- $\Omega_\varepsilon = \Omega^- \cup \widetilde{\Omega}_\varepsilon \cup \Omega^+ \cup S^- \cup S^+$, where $S^\pm = \Omega' \times \{\pm L\}$.
- $\mathcal{A}_\varepsilon^{d_1,\ldots,d_m}$ is the minus Laplacian on Ω_ε subject to the Dirichlet conditions on $\partial\Omega \setminus \cup_{k=1}^m S_{k,\varepsilon}$ and the Neumann conditions on $S_{k,\varepsilon}$.

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One has $\sigma_{\mathrm{ess}}(\mathcal{A}_\varepsilon^{d_1,\dots,d_m}) = [\Lambda, \infty)$, where $\Lambda = \lambda_1(-\Delta_{\Omega''}^D) > 0$.

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We introduce the functions $\mathcal{F}_k : \mathbb{R}_+ \to \mathbb{R}_+$ via

$$\mathcal{F}_k(t) = \begin{cases} \left(t\dfrac{4|B_k|}{\mathrm{cap}(D_k)}\right)^{\frac{1}{n-2}}, & n \ge 3,\\ t\dfrac{2|B_k|}{\pi}, & n = 2.\end{cases}$$

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Theorem

Let $\widetilde{\gamma}_k$, $k \in \{1, \dots, m\}$ be arbitrary numbers satisfying

$$0 < \widetilde{\gamma}_1 < \widetilde{\gamma}_2 < \cdots < \widetilde{\gamma}_m < \Lambda.$$

Then $\delta > 0$ there exist $\varepsilon > 0$ and $\widetilde{d}_k \in (\mathcal{F}_k(\widetilde{\gamma}_k) - \delta, \mathcal{F}_k(\widetilde{\gamma}_k) + \delta)$, $k \in \{1, \dots, m\}$ such that

$$\sigma(\mathcal{A}_\varepsilon^{\widetilde{d}_1,\dots,\widetilde{d}_m}) \cap [0, \Lambda) = \bigcup_{k=1}^{m} \{\widetilde{\gamma}_k\}.$$

The eigenvalues $\widetilde{\gamma}_k$ are simple.

Thank you for your attention!