Superintegrability in non-inertial reference frames with conic defects

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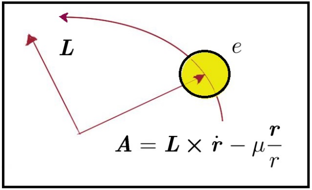
Based on:

1. L. Inzunza and M. S. Plyushchay, JHEP 2021, 165 (2021) [arXiv:2012.04613 [hep-th]].
2. L. Inzunza and M. S. Plyushchay, Phys. Rev. D 103, 106004 (2021) [arXiv:2103.07752 [quant-ph]].
3. L. Inzunza and M. S. Plyushchay, JHEP 2022, 179 (2022) [arXiv:2109.05161[hep-th]].

Introduction

Symmetries control peculiarities of physical systems at the classical and quantum level[4,5,6,7].

Conserved quantities $\rightarrow$ Conserved operators



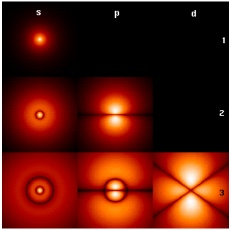


Figure: On the left: Hidden symmetries in the classical Kepler problem. $\boldsymbol{A}$ is the so-called Laplace-Runge-Lentz vector. On the right: probability densities through the xz-plane for the electron at different quantum numbers (l, across top; n, down side; $m = 0$) (Wikipedia:Hydrogen atom).

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4. W. Pauli, Z Phys 36: 336-363 (1926).
5. J. J. Sakurai (1985). Modern quantum mechanics.
6. J. de Boer, F. Harmsze and T. Tjin, Phys. Rept. **272** (1996), 139-214 [arXiv:hep-th/9503161 [hep-th]].
7. M. Cariglia, Rev. Mod. Phys. **86**, 1283 (2014) [arXiv:1411.1262[math-th]].

Classical dynamics in curved spaces

Given a space-time metric of the form

$$ds^2 = -c^2 dt^2 + g_{ij}(x) dx^i dx^j \; ,$$

one can construct the relativistic action

$$S_R = -mc \int \sqrt{c^2 dt^2 - g_{ij}(x) dx^i dx^j} \; .$$

Symmetries of $S_R \rightarrow$ Killing vectors and tensors associated to ds^2 [8,9].
The non-relativistic limit:

$$S = \int \frac{m}{2} g_{ij}(x) \frac{dx^i}{dt} \frac{dx^j}{dt} \; .$$

Symmetries of $S \rightarrow$ non-relativistic limits [10,11] of the Killing vectors and Killing tensors associated to ds^2.

8. M. Cariglia, Rev. Mod. Phys. **86**, 1283 (2014) [arXiv:1411.1262[math-th]].
9. M. Crampin, Reports on Math. Phys **20** (1984) 31.
10. D. T. Son, Phys. Rev. D **78** (2008) 046003 [arXiv: 0804.3972 [hep-th]].
11. A. Bagchi and R. Gopakumar, JHEP **0907** (2009) 037[arXiv: 0902.1385 [hep-th]].

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Rotating background [12,13,14]

The metric of the $(3+1)$ Minkowski vacuum subjected to a uniform rotation Ω along the z axis in polar coordinates is given by

$$ds^2 = -c^2 dt + d\rho^2 + \rho^2(d\varphi + \Omega dt)^2 + dz^2\,.$$

The non-relativistic Lagrangian

$$L = L_\Omega + \tfrac{m}{2}\dot{z}^2\,, \qquad L_\Omega = \tfrac{m}{2}(\dot{\rho}^2 + \rho^2(\varphi^2 + \Omega)^2)\,,$$

formally relates with the flat Lagrangian by $\varphi(t) \to \varphi(t) + \Omega t$.

In Cartesian coordinates ($g_{ij} = \delta_{ij}$)

$$L_\Omega = g_{ij}(\tfrac{m}{2}\dot{x}^i\dot{x}^i + \tfrac{m}{2}\Omega^2 x^i x^i + \tfrac{q_{GM}}{c}\dot{x}^i A^i)\,, \quad A_i = \tfrac{\Omega c}{2}\epsilon^{ij}x^j\,.$$

Gravitoelectromagnetism

$B_G = \epsilon_{ij}\partial_i A_j = -\Omega c$ is interpret as constant Gravitomagnetic field.
$q_{GM} = -2m$ is the Gravitomagnetic charge.

12. T. Kibble, J. Phys. A **9** (1976) 1387.
13. A. Vilenkin, Phys. Rev. D **23** (1981) 852.
14. A. Vilenkin, Phys. Rept. **121** (1985) 263.

Free motion

The geodesic Hamiltonian in polar coordinates

$$H_\Omega = H_0 - \Omega p_\varphi \,, \qquad H_0 = \tfrac{1}{2m} g^{ij} p_i p_i \,, \qquad p_\varphi = x^1 p_2 - x^2 p_1 \,,$$

shows that the energy is not restricted from below (#).

The equations of motions are solved by a rotation of the straight line,

$$x_+(t) = x^1 + i x^2 = \rho e^{i\varphi} = e^{-i\Omega t}(Bt + C) \,, \qquad B, C \in \mathbb{C} \,.$$

If the minimum of the trajectory occurs at t_*, then, the value of $x_+(t_*) = \rho_* e^{i\varphi_*}$ is controlled by the the integrals of motion.

$$\rho_* = \tfrac{|p_\varphi|}{\sqrt{2mH_0}} \,, \qquad e^{\pm i\varphi_*} \approx I_\pm = \exp\Big(\pm \tfrac{\Omega \rho p_\rho}{2H_0}\Big)(p_1 \pm i p_2) \,.$$

The system is ill-defined at the quantum level (#).

Harmonically confined motion

We supplement L_Ω with an harmonic potential

$$L_\gamma = L_\Omega - \tfrac{1}{2} m \gamma^{-2} \Omega^2 g_{ij} x^i x^j \,. \qquad \gamma \in \mathbb{R}.$$

When introducing a new frequency ω by the equation $\gamma\omega = -\Omega$, the Hamiltonian takes the form

$$H_\gamma = H_{\text{osc}} + \gamma\omega p_\varphi \,, \qquad H_{\text{osc}} = H_0 + \tfrac{1}{2} m \omega^2 g_{ij} x^i x^j .$$

In terms of *"spherical"* ladder operators classical analogs,

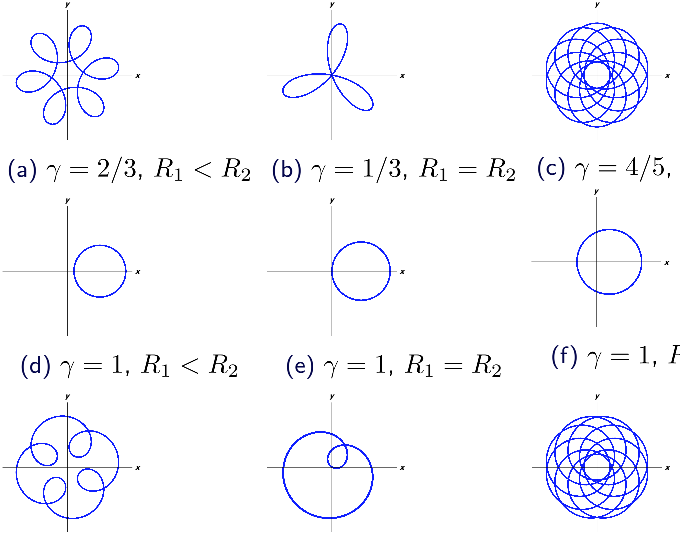
$$\begin{aligned} b_j^\pm &= \tfrac{1}{\sqrt{2}}(a_1^\pm \mp i(-1)^j a_2^\pm) \,, \quad \{b_j^\pm, b_k^\mp\} = -i\delta_{jk} \,, \\ a_i^\pm &= \sqrt{\tfrac{m\omega}{2}}(x_i \pm \tfrac{i}{m\omega} p_i) \,, \end{aligned}$$

the Hamiltonian and the equations of motion looks like

$$\begin{aligned} &H_\gamma = \omega(\ell_1 b_1^+ b_1^- + \ell_2 b_2^+ b_2^-) \,, \quad \ell_1 = 1+\gamma \,, \quad \ell_2 = 1-\gamma \,, \\ &\dot{b}_i^\pm = \{b_i^\pm, H\} = \pm i\omega\ell_i b_i^\pm \qquad \Rightarrow \qquad b_i^\pm(t) = e^{\pm i\omega\ell_i t} b_i^\pm(0) \,. \end{aligned}$$

In terms of $x_+(t)$:

$$x_+(t) = R_1 e^{i\vartheta_1} e^{i\omega\ell_1 t} + R_2 e^{-i\vartheta_2} e^{-i\omega\ell_2 t} \,.$$



(a) $\gamma = 2/3,\ R_1 < R_2$ (b) $\gamma = 1/3,\ R_1 = R_2$ (c) $\gamma = 4/5,\ R_1 > R_2$

(d) $\gamma = 1,\ R_1 < R_2$ (e) $\gamma = 1,\ R_1 = R_2$ (f) $\gamma = 1,\ R_1 > R_2$

(g) $\gamma = 3/2,\ R_1 < R_2$ (h) $\gamma = 3,\ R_1 = R_2$ (i) $\gamma = 5/4,\ R_1 > R_2$

Figure: Roulettes: Trajectories for some rational values of γ. In cases b), e) and h), $p_\varphi = 0$ and trajectories pass through the origin.

Integrability in rotating harmonic trap

The classical ladder operators in polar coordinates are

$$b_j^- = \tfrac{1}{2}e^{(-1)^j i\varphi}(\sqrt{m\omega}\rho - (-1)^j \tfrac{p_\varphi}{\sqrt{m\omega}\rho} + \tfrac{ip_\rho}{\sqrt{m\omega}})\,.$$

Integrals of motion appears for $\gamma \in \mathbb{Q}$.

$$\begin{array}{lll} \gamma = \frac{s_2-s_1}{s_1+s_2} & |\gamma| \leq 1 & \mathcal{L}^{\pm}_{s_1,s_2} = (b_1^{\pm})^{s_1}(b_2^{\mp})^{s_2} \\ \gamma = \frac{s_2+s_1}{s_1-s_2} & |\gamma| > 1 & \mathcal{J}^{\pm}_{s_1,s_2} = (b_1^{\pm})^{s_1}(b_2^{\pm})^{s_2} \end{array}\,.$$

Poisson brackets:

$$\begin{array}{ll} \{H_{\mathsf{osc}}, \mathcal{L}^{\pm}_{s_1,s_2}\} = \mp i\omega(s_1-s_2)\mathcal{L}^{\pm}_{s_1,s_2} & \{p_\varphi, \mathcal{L}^{\pm}_{s_1,s_2}\} = i(s_1+s_2)\mathcal{L}^{\pm}_{s_1,s_2} \\ \{H_{\mathsf{osc}}, \mathcal{J}^{\pm}_{s_1,s_2}\} = \mp i\omega(s_1+s_2)\mathcal{J}^{\pm}_{s_1,s_2} & \{p_\varphi, \mathcal{J}^{\pm}_{s_1,s_2}\} = i(s_1-s_2)\mathcal{J}^{\pm}_{s_1,s_2} \end{array}\,.$$

and so ($I^{\pm}_{s_1,s_2}$ equal to $\mathcal{L}^{\pm}_{s_1,s_2}$ or $\mathcal{J}^{\pm}_{s_1,s_2}$),

$$\{H_\gamma, I^{\pm}_{s_1,s_2}\} = \{H_{\mathsf{osc}}, I^{\pm}_{s_1,s_2}\} + \gamma\omega\{p_\varphi, I^{\pm}_{s_1,s_2}\} = 0\,.$$
$$\{I^{+}_{s_1,s_2}, I^{-}_{s_1,s_2}\} = P(H_{\mathsf{osc}}, p_\varphi)\,.$$

Quantum picture

Canonical quantization

$$
\begin{aligned}
&b_j^{\pm}(x,p) \to \hat{b}_j^{\pm} = \hbar^{-\frac{1}{2}} b_j^{\pm}(\hat{x},\hat{p})\,, \qquad [\hat{b}_i^{\mp},\hat{b}_j^{\pm}] = \delta_{ij}\,,\\
&\hat{H}_\gamma = \hbar\omega(\ell_1 \hat{b}_1^+\hat{b}_1^- + \ell_2 \hat{b}_2^+\hat{b}_2^- + 1)\,, \quad \hat{p}_\varphi = \hbar(\hat{b}_1^+\hat{b}_1^- - \hat{b}_2^+\hat{b}_2^-)\,.
\end{aligned}
$$

Stationary eigenstates Ψ_{n_1,n_2} with $n_1, n_2 = 0, \ldots,$ that satisfy

$$
\begin{aligned}
&\hat{b}_1^-\Psi_{n_1,n_2} = \sqrt{n_1}\Psi_{n_1-1,n_2}\,, \qquad \hat{b}_1^+\Psi_{n_1,n_2} = \sqrt{n_1+1}\Psi_{n_1+1,n_2}\,,\\
&\hat{b}_2^-\Psi_{n_1,n_2} = \sqrt{n_2}\Psi_{n_1,n_2-1}\,, \qquad \hat{b}_2^+\Psi_{n_1,n_2} = \sqrt{n_2+1}\Psi_{n_1,n_2-1}\,.
\end{aligned}
$$

Eigenvalue equations are:

$$
\begin{aligned}
&\hat{H}_\gamma\Psi_{n_1,n_2} = E^{(\gamma)}_{n_1,n_2}\Psi_{n_1,n_2}\,, \qquad E^{(\gamma)}_{n_1,n_2} = \hbar\omega(\ell_1 n_1 + \ell_2 n_2 + 1)\\
&\hat{p}_\varphi\Psi_{n_1,n_2} = \hbar(n_1 - n_2)\Psi_{n_1,n_2}\,.
\end{aligned}
$$

In coordinate representation, Ψ_{n_1,n_2} is a linear combination of Hermite polynomials times $e^{-\frac{m\omega}{2\hbar}(x_1^2+x_2^2)}$.

Degeneracy

For $\gamma = \frac{s_2 - s_1}{s_1 + s_2}$
$$\hat{\mathcal{L}}^{+}_{n_1,n_2}\Psi_{n_1,n_2} = \sqrt{\frac{n_2!}{n_1!}\frac{\Gamma(n_1+s_1+1)}{\Gamma(n_2-s_2+1)}}\Psi_{n_1+s_1,n_2-s_2}$$
$$\hat{\mathcal{L}}^{-}_{n_1,n_2}\Psi_{n_1,n_2} = \sqrt{\frac{n_1!}{n_2!}\frac{\Gamma(n_2+s_2+1)}{\Gamma(n_1-s_1+1)}}\Psi_{n_1-s_1,n_2+s_2}$$

- $E^{(\gamma)}_{s_1,s_2}$ is bounded from below.
- $E^{(\gamma)}_{s_1,s_2}$ is invariant under $n_1 \to n_1 \pm s_1$ and $n_2 \to n_2 \mp s_2$.
- $|\gamma| < 1$, there are finite number of eigenstates for each energy level.
- For $|\gamma| = 1$, each energy level is infinite degenerate (Landau problem).

For $\gamma = \frac{s_2 + s_1}{s_2 - s_2}$
$$\hat{\mathcal{J}}^{-}_{n_1,n_2}\Psi_{n_1,n_2} = \sqrt{\frac{n_1!n_2!}{\Gamma(n_1-s_1+1)\Gamma(n_2-s_2+1)}}\Psi_{n_1-s_1,n_2-s_2}$$
$$\hat{\mathcal{J}}^{+}_{n_1,n_2}\Psi_{n_1,n_2} = \sqrt{\frac{\Gamma(n_1+s_1+1)\Gamma(n_2+s_2+1)}{n_1!n_2!}}\Psi_{n_1+s_1,n_2+s_2}$$

- $E^{(\gamma)}_{s_1,s_2}$ is not bounded from below.
- $E^{(\gamma)}_{s_1,s_2}$ is invariant under $n_1 \to n_1 \pm s_1$ and $n_2 \to n_2 \mp s_2$.
- Each energy level has infinite degeneracy.

A cosmic string background [12,13,14]

The metric produced by a cosmic string is

$$dS^2 = -c^2 dt^2 + ds^2 \,, \quad ds^2 = \left(1 - \frac{8\mu G}{c^2} \ln\left(\frac{r}{r_0}\right)\right)(dr^2 + r^2 d\varphi^2) \,,$$

G: Newton constant.
c: Speed of light.
μ: Linear mass density of the cosmic string.
r_0: Cosmic string radius.
By introducing the new coordinate

$$r'^2 = \left(1 - \tfrac{8\mu G}{c^2} \ln\left(\tfrac{r}{r_0}\right)\right) r^2 \,, \quad \alpha^2 dr'^2 = \left(1 - \tfrac{8\mu G}{c^2} \ln\left(\tfrac{r}{r_0}\right)\right) dr^2 \,,$$

$$\alpha = \frac{1}{1 - \frac{4\mu G}{c^2}} > 0 \,,$$

one gets (renaming $r' \to r$)

$$ds^2 = \alpha^2 dr^2 + r^2 d\varphi^2 \,.$$

12. T. Kibble, J. Phys. A **9** (1976) 1387.
13. A. Vilenkin, Phys. Rev. D **23** (1981) 852.
14. A. Vilenkin, Phys. Rept. **121** (1985) 263.

The conical metric

$$ds^2 = \alpha^2 dr^2 + r^2 d\varphi^2\,.$$

$$g_{ij} = \delta_{ij} + (\alpha^2 - 1)\frac{1}{(x^1)^2+(x^2)^2}\begin{pmatrix} (x^1)^2 & x^1x^2 \\ x^1x^2 & (x^2)^2 \end{pmatrix},$$

$\alpha > 1$:

- Reduction from $ds^2_E = dx^2 + dy^2 + dz^2$ to the conical surface $z = \lambda_E r$. $\alpha = 1 + \lambda_E^2$.
- $\alpha > 1$ implies $\mu > 0$.

$0 < \alpha < 1$:

- Reduction from $ds^2_M = -c^2 d\tau^2 + dx^2 + dy^2$ to the surface $\tau = \lambda_M r$ (non-causal), $\alpha = 1 - \lambda_M^2$.
- $0 < \alpha < 1$ implies $\mu < 0$. Acquires a different interpretation in condensed matter [15,16].

ds^2 is formally obtained from the plane metric be means of the transformation $\rho \to \alpha\rho$ and $\varphi \to \alpha^{-1}\varphi$.

15. G. E. Volovik, (Oxford Science Publications, 2003).
16. N. S. Manton, J. Phys. A **50** (2017) 125403 [arXiv: 1612.06710 [hep-th]].

Non-relativistic motion in a cone

Consider the systems

$$I^{(\alpha)} = \int L^{(\alpha)} dt\,, \qquad L^{(\alpha)} = \frac{m}{2} g_{ij} \frac{dx_i}{dt}\frac{dx_j}{dt} = \frac{m}{2}\left(\alpha^2 \dot{r}^2 + r^2 \dot{\varphi}^2\right)\,,$$

$$I^{(\alpha)}_{\mathsf{osc}} = \int L^{(\alpha)}_{\mathsf{osc}} dt, \quad L^{(\alpha)}_{\mathsf{osc}} = \frac{m}{2}\left(\alpha^2 \dot{r}^2 + r^2 \dot{\varphi}^2\right) - \frac{m\omega^2\alpha^2}{2} r^2\,,$$

where g_{ij} is the conical metric.
The Hamiltonian are [17,18]:

$$H^{(\alpha)} = \frac{1}{2m}\left(\frac{p_r^2}{\alpha^2} + \frac{p_\varphi^2}{r^2}\right)\,, \quad H^{(\alpha)}_{\mathsf{osc}} = H^{(\alpha)} + \frac{m\omega^2\alpha^2}{2} r^2\,.$$

Notable!

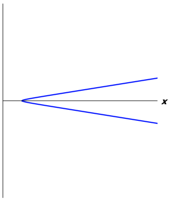
The local canonical transformation

$$r \to \alpha r\,, \qquad p_r \to \frac{p_r}{\alpha}\,, \qquad \varphi \to \frac{\varphi}{\alpha}\,, \qquad p_\varphi \to \alpha p_\varphi\,,$$

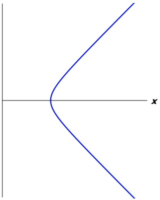
applied to the Hamiltonians in the Euclidian plane gives us $H^{(\alpha)}$ and $H^{(\alpha)}_{\mathsf{osc}}$.

17. G.'t Hooft, Commun. Math. Phys. **117** (1988) 685.
18. S. Deser and R. Jackiw, Comm. Math. Phys **118** (1988) 495.

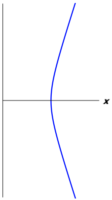
Solutions of the equations of motion



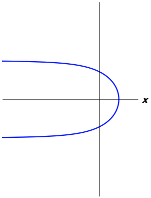
(a) $\alpha = 1/10$



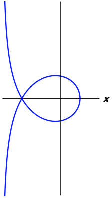
(b) $\alpha = 1/2$



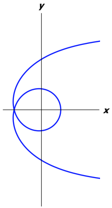
(c) $\alpha = 4/5$



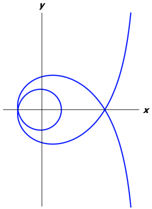
(d) $\alpha = 2$



(e) $\alpha = 3$



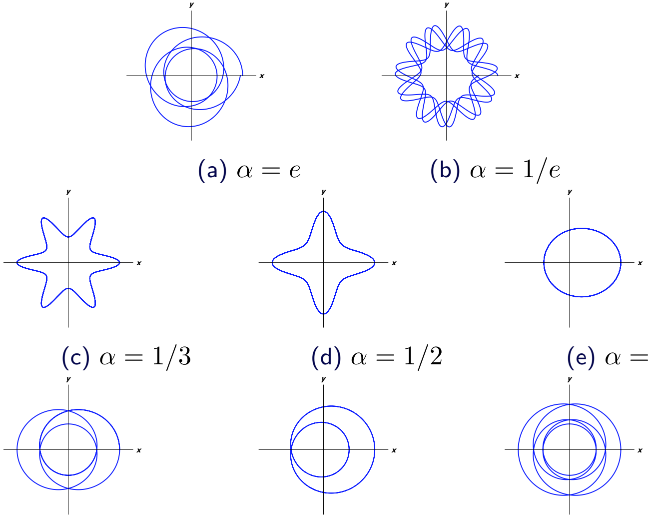
(f) $\alpha = 4$



(g) $\alpha = 5$

Figure: Solutions of the equations of motion for different values of α.

Solutions of the equations of motion



(a) $\alpha = e$ (b) $\alpha = 1/e$

(c) $\alpha = 1/3$ (d) $\alpha = 1/2$ (e) $\alpha = 2$

(f) $\alpha = 3$ (g) $\alpha = 4$ (h) $\alpha = 5$

Figure: Only when α is rational one has closed trajectories.

A rotating conic background

After applying the transformation $\rho \to \alpha\rho$ and $\varphi \to \alpha^{-1}\varphi$ (and $\Omega \to \alpha\Omega$) in to the rotating vacuum metric we get

$$ds^2 = -c^2dt^2 + \alpha^2 d\rho^2 + \rho^2(d\varphi + \alpha\Omega dt)^2\,.$$

The non-relativistic dynamics is obtained by the same transformation:

Geometric background	Phase space functions
Plane	$f(\rho, \varphi, p_\rho, p_\varphi)$
Cone	$f(\alpha\rho, \alpha^{-1}\varphi, \alpha^{-1}p_\rho, \alpha p_\varphi)$

Planar solutions transform as $x_\pm(t) \to X_{\alpha,\pm}(t) = \alpha\rho(t)e^{\frac{i\varphi(t)}{\alpha}}$.
The solution we are looking

$$x_{\alpha,\pm} = \rho(t)e^{i\varphi(t)} = \alpha^{-1}|X_{\alpha,\pm}(t)|^{1-\alpha}(X_{\alpha,\pm})^\alpha$$

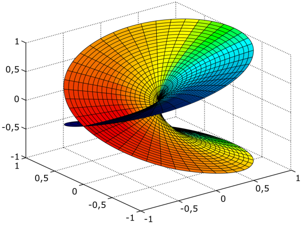
The domain of $(X_{\alpha,\pm})^\alpha$ is a Riemann surface depending on the value of α.

Drawing in a Riemann surface (RS)

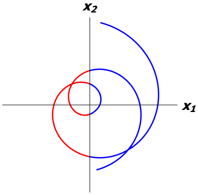
Consider the complex functions:

$$z = |z| e^{i(\vartheta + 2\pi n)}, \qquad f_\alpha(z) = z^\alpha = |z|^\alpha e^{i\alpha\vartheta} e^{i2\pi n\alpha}$$

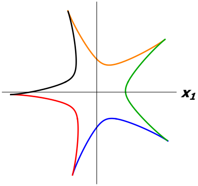
1. $\alpha \in \mathbb{Z}^+$ $f_\alpha(z) \in \mathbb{C}$.
2. $\alpha = \frac{q}{k}$,$f_\alpha(z) \in$ RS of k sheets.
3. $\alpha \in \mathbb{I}$, $f_\alpha(z) \in$ RS with infinite sheets (like $\ln(z)$).



(a) $\sqrt{z}$.



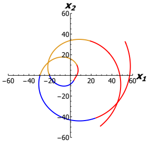
(b) Trajectory F.P $\alpha = 1/2$.



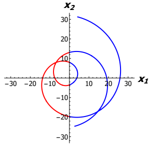
(c) Trajectory H.O $\alpha = 3/5$, $\gamma = 1/3$.

Figure: Wikipedia: Riemann surfaces.

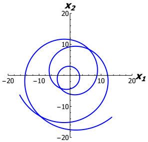
Classical trajectories: free case



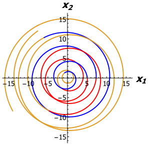
(a) $\alpha = 1/3,\ \dot{\varphi}_* < 0.$



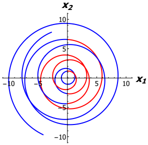
(b) $\alpha = 1/2,\ \dot{\varphi}_* < 0.$



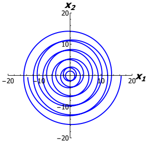
(c) $\alpha = 1,\ \dot{\varphi}_* < 0.$



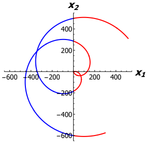
(d) $\alpha = 4/3,\ \dot{\varphi}_* < 0.$



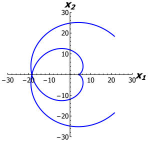
(e) $\alpha = 3/2,\ \dot{\varphi}_* < 0.$



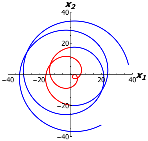
(f) $\alpha = 2,\ \dot{\varphi}_* < 0.$



(g) $\alpha = 1/2,\ \dot{\varphi}_* > 0.$

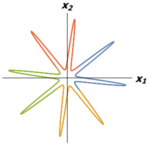


(h) $\alpha = 1,\ \dot{\varphi}_* = 0.$

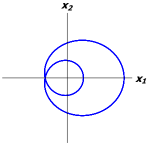


(i) $\alpha = 3/2,\ \dot{\varphi}_* > 0.$

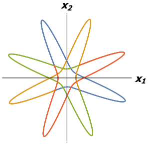
Classical trajectories: Harmonic oscillator case



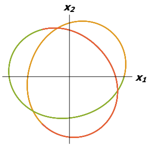
(a) $\gamma = 0$, $\alpha = 1/4$.



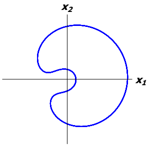
(b) $\gamma = 0$, $\alpha = 4$.



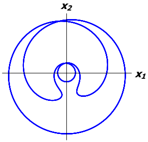
(c) $\gamma = 0$, $\alpha = 3/4$.



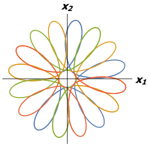
(d) $\gamma = 1$, $\alpha = 2/3$.



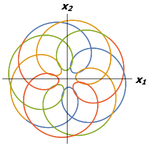
(e) $\gamma = 1$, $\alpha = 3$.



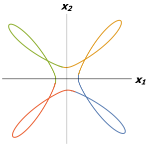
(f) $\gamma = 1$, $\alpha = 7$.



(g) $\gamma = 1/2$, $\alpha = 3/4$.



(h) $\gamma = 2$, $\alpha = 3/4$.



(i) $\gamma = 1/3$, $\alpha = 3/4$.

Integrals of motion

The complex quantities

$$b^{-}_{\alpha,j} = \tfrac{1}{2}e^{(-1)^j i\frac{\varphi}{\alpha}}\left(\alpha\sqrt{m\omega}\rho - (-1)^j\frac{p_\varphi}{\sqrt{m\omega}\rho} + \frac{ip_\rho}{\alpha\sqrt{m\omega}}\right),$$

$$\mathcal{L}^{\pm}_{\alpha,s_1,s_2} = (b^{\pm}_{\alpha,1})^{s_1}(b^{\mp}_{\alpha,2})^{s_2}\,, \qquad \mathcal{J}^{\pm}_{\alpha,s_1,s_2} = (b^{\pm}_{\alpha,1})^{s_1}(b^{\pm}_{\alpha,2})^{s_2}\,,$$

are not well defined on phase-space for $\alpha \neq 1$.
For $\gamma = \frac{s_2-s_1}{s_2+s_2}$ and $\alpha = q/k$ we have instead

$$\mathscr{L}^{(\epsilon)\pm}_{\alpha,s_1,s_2} = (\mathcal{L}^{\pm}_{\alpha,s_1,s_2})^{\epsilon}\,, \quad \epsilon = \begin{cases} r & \text{if} \quad q = r(s_1+s_2)\,, \quad r = 1,2,\ldots, \\ \\ q & \text{if} \quad q \neq r(s_1+s_2)\,, \end{cases}$$

For $\gamma = \frac{s_2+s_1}{s_2-s_2}$ and $\alpha = q/k$ we have

$$\mathscr{J}^{(\delta)\pm}_{\alpha,s_1,s_2} = (\mathcal{J}^{\pm}_{\alpha,s_1,s_2})^{\delta}\,, \quad \delta = \begin{cases} r' & \text{if} \quad q = r'|s_2-s_1|\,, \quad r' = 1,2,\ldots, \\ \\ q & \text{if} \quad q \neq r'|s_2-s_1|\,, \end{cases}$$

General comments abut the quantum cases

$$\hat{H}_{\gamma}^{(\alpha)} = -\frac{\hbar^2}{2m}\left(\frac{1}{\alpha^2\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi}\right) + \frac{\alpha^2 m\omega^2}{2}\rho^2 - i\gamma\omega\hbar\frac{\partial}{\partial\varphi}\,,$$

whose eigenstates and spectrum are given by

$$\Psi_{n_\rho,\ell}^{(\alpha,\gamma)}(\rho,\varphi) = \left(\frac{m\omega\alpha^2}{\hbar}\right)^{\frac{1}{2}}\sqrt{\frac{n_\rho!}{2\pi\alpha\Gamma(n_\rho+\alpha|\ell|+1)}}\,\zeta^{\alpha|\ell|}L_{n_\rho}^{(\alpha|\ell|)}(\zeta^2)e^{-\frac{\zeta^2}{2}+i\ell\varphi}\,,$$

$$\zeta = \sqrt{\frac{m\alpha^2\omega}{\hbar}}\rho\,,\quad E_{n_\rho,\ell}^{(\alpha,\gamma)} = \hbar\omega(2n_\rho + \alpha(1+\gamma\,\mathsf{sign}(\ell))|\ell| + 1)\,,$$

$$n_r = 0,1,\ldots\,,\qquad \ell = 0,\pm1,\ldots\,.$$

General proprieties

1. New degeneracy when $\alpha,\gamma\in\mathbb{Q}$.
2. Quantum operators $\hat{\mathscr{L}}_{\alpha,s_1,s_2}^{(\epsilon)\pm}$ $\hat{\mathscr{J}}_{\alpha,s_1,s_2}^{(\epsilon)\pm}$ are well defined on Hilbert space only when $\alpha = n$.

Conclusion and outlook

Geometry and reinterpretations

i) Non-inertial effects reinterpret as a magnetic fields.

ii) Conic background and cosmic strings.

iii) Rotating conic background and spinning cosmic strings ds^2_{sp} [19]?.

With $d\varphi \to \frac{1}{2}(d\varphi - 2\alpha\Omega dt)$, $dt \to dt + \frac{1}{2\alpha\Omega}d\varphi$ (periodic time), and $J = \frac{c^4}{8\alpha\Omega G}$, the rotating conic metric transform in,

$$ds^2_{\mathsf{sp}} = -c^2(dt + 4c^{-4}GJd\varphi)^2 + \alpha^2 d\rho^2 + \rho^2 d\varphi^2\,.$$

Geometry and classical harmonic oscillator dynamics

i) Free parameters in geometry (α and Ω) may imply special characteristics of the paths.

ii) For rational α and $|\gamma| \in \mathbb{Q}$ (Commensurability of frequencies) exist higher order integrals of motion and closed paths (they are correlated).

19.P. O. Mazur, Phys. Rev. Lett. 57 (1986) 929.

Geometry and quantum harmonic oscillator dynamics

i) For $\alpha \in \mathbb{N}$ and $|\gamma| \in \mathbb{Q}$, higher order integrals operator describe the spectrum.

ii) *Multiparticle generalization:* Bose Einstein condensation in the cone.

Open questions:

i) Interpretation of the higher-order integrals of motion in geometric terms (Killing tensors).

ii) Investigate the quantum anomaly under the light of exotic boundary conditions [20].

iii) To study different classical and quantum systems in this an other geometries.

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20. B.S. Kay, and U.M. Studer, Commun. Math. Phys. **139** (1991) 103.


Thank you very much for your attention !

A multi particle generalization

1. **Gross-Pitaevskii formalism**

Hamiltonian of N particles in the cone geometry

$$\hat{\mathcal{H}} = \sum_{n=1}^{N} H_\gamma^{(\alpha)}(\boldsymbol{r}_i) + \mathcal{U}(r_1, \ldots, r_1)\,, \qquad \mathcal{U} = \sum_{i<j} U(\boldsymbol{r}_i, \boldsymbol{r}_j).$$

In the Harthree-Fock approximation: $U = \frac{4\pi\hbar a_s}{m}\delta(\boldsymbol{r}_i - \boldsymbol{r}_j)$ with $a_s \sim 0$.
Using low temperature approximation (μ chemical potential)

$$\mu\psi(\boldsymbol{r}) = \left(\hat{H}_\gamma^{(\alpha)} + \frac{4\pi\hbar a}{m}|\psi|^2\right)\psi(\boldsymbol{r})\,, \qquad \psi = \langle\hat{\Psi}\rangle.$$

We take ψ as a linear combination of each solutions associated to the ground state of $\hat{H}_\gamma^{(\alpha)}$ such that

$$I = \frac{4\pi\hbar a_s}{m}\int \alpha\rho d\rho d\varphi|\psi|^4 \sim 0\,.$$

Landau: $\gamma = 1\,, \quad \psi^{(\alpha)} = \sqrt{N}\sum_{l=0}^{l_{\mathsf{cut}}} c_l \Psi_{0,l}^{(\alpha,1)} \quad \sqrt{N}\sum_{l=0}^{l_{\mathsf{cut}}} c_l c_l^* = 1 \quad .$

Vortices

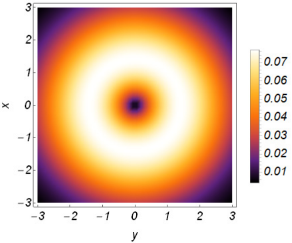
2. **The zeros of $\psi^{(\alpha)}$ are interpret as vortices.**

For testing we chose $\ell_{\text{cut}} = 2$ and $\frac{m\omega}{\hbar} = 1$.

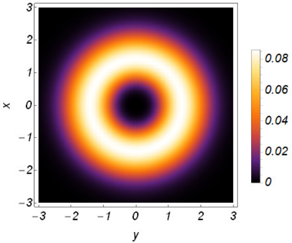
$$\psi^{(\alpha)} = \sqrt{\frac{\alpha N}{\pi(|a_0|^2+|a_1|^2+|a_2|^2)}} \left(a_0 + \frac{a_1(\alpha\rho)^\alpha}{\sqrt{\Gamma(\alpha+1)}} e^{i\varphi} + \frac{a_2(\alpha\rho)^{2\alpha}}{\sqrt{\Gamma(\alpha+1)}} e^{i2\varphi} \right) e^{-\frac{\alpha^2\rho^2}{2}} .$$

α	a_0	a_1	a_2	I_{a_0,a_1,a_2}
$1/4$	0	0	1	$1/(4\pi) \approx 0.025$
$1/3$	0	0	1	$\Gamma(7/6)/(6\pi^{\frac{3}{2}}\Gamma(5/3)) \approx 0.031$
$1/2$	0	0	1	$1/(8\pi) \approx 0.040$
1	0	0	1	$3/(16\pi) \approx 0.060$
2	1	0	$\sqrt{112/19} \approx 2.428$	$33/(131\pi) \approx 0.080$
3	1	0	$\sqrt{992/199} \approx 2.233$	$115/(397\pi) \approx 0.092$
4	1	0	$\sqrt{32512/6179} \approx 2.294$	$12866/(38691\pi) \approx 0.106$

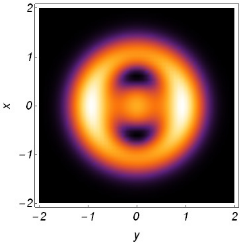
Table: The values of parameters that minimize I_{a_0,a_1,a_2} for different values of α, where a_i are assumed to be real numbers (the imaginary part only contributes with a global sign in this particular case). Note that the value of I_{a_0,a_1,a_2} grows with the increasing value of α.



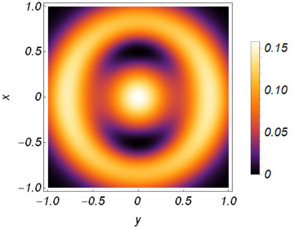
(a) $\alpha = 1/2$



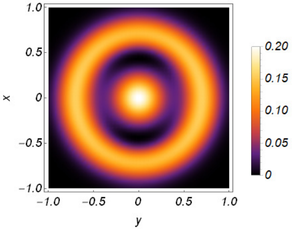
(b) $\alpha = 1$



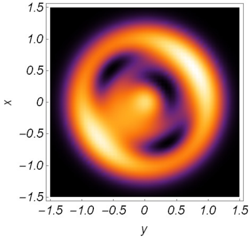
(c) $\alpha = 2$



(d) $\alpha = 3$



(e) $\alpha = 4$



(f) $\alpha = 3$