

Non-self-adjoint relativistic point interaction in one dimension

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General relativistic point interactions

$$\mathcal{D} := -ic\sigma_1 \frac{d}{dx} + mc^2\sigma_3, \quad \mathcal{D}^{\mathbb{A}} := \mathcal{D} + c\mathbb{A}|\delta(x)\rangle\langle\delta(x)|,$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbb{A} \in \mathbb{C}^{2,2}, m \geq 0, c > 0.$$

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$$\langle\delta(x)|\psi\rangle\delta(x)\varphi := \frac{\psi(0+) + \psi(0-)}{2}\varphi(0) \in \mathbb{C}^2.$$

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Definition (Relativistic point interactions)

$$\text{Dom}(D^{\mathbb{A}}) = \{\psi \equiv \psi_- \oplus \psi_+ \in H^1(\mathbb{R}_-; \mathbb{C}^2) \oplus H^1(\mathbb{R}_+; \mathbb{C}^2) \mid (1) \text{ holds}\},$$

$$D^{\mathbb{A}}\psi = \mathcal{D}\psi_- \oplus \mathcal{D}\psi_+.$$

[Seba 89]

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$D^{\mathbb{A}}$ self-adjoint extensions where $\mathbb{A} = \mathbb{A}^*$.

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What about non-hermitian \mathbb{A} ?

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$$V_\varepsilon = \frac{1}{\varepsilon^2} |v(x/\varepsilon)\rangle \langle v(x/\varepsilon)| \quad [\text{Seba 89}].$$

Free Dirac operator

$$(D_0\psi)(x) = (\mathcal{D}\psi)(x), \forall x \in \mathbb{R},$$

$$\text{Dom } D_0 = H^1(\mathbb{R}) \otimes \mathbb{C}^2.$$

$$\sigma(D_0) = \{(-\infty, -mc^2] \cup [mc^2, +\infty)\} =: \mathbb{R}_{mc^2}.$$

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The resolvent

$$R_z(x, y) = \frac{i}{2c} (\mathbb{Z}(z) + \text{sgn}(x - y)\sigma_1) e^{ik(z)|x-y|}.$$

$$\mathbb{Z}(z) = \begin{pmatrix} \zeta(z) & 0 \\ 0 & \zeta(z)^{-1} \end{pmatrix},$$

$$\zeta(z) = \frac{z + mc^2}{ck(z)}, ck(z) = \sqrt{z^2 - (mc^2)^2}, \text{Im} k(z) \geq 0.$$

The resolvent of $D^{\mathbb{A}}$

$$R_z^{\mathbb{A}}(x, y) = R_z(x, y) - cR_z(x, 0)(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))^{-1}\mathbb{A}R_z(0, y).$$

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Theorem

Let \mathbb{A} and $z \in \mathbb{C} \setminus \mathbb{R}_{mc^2}$ such that

$$(I + \frac{i}{2}\mathbb{A}\mathbb{Z}(z))$$

is invertible. Then

$$(D_0 + cV_\varepsilon\mathbb{A} - z)^{-1} \xrightarrow[\varepsilon \rightarrow 0]{u} (D^{\mathbb{A}} - z)^{-1}.$$

Spectrum

Theorem

$$\sigma(D^{\mathbb{A}}) \setminus \mathbb{R}_{mc^2} = \sigma_p(D^{\mathbb{A}}).$$

$z \in \mathbb{C} \setminus \mathbb{R}_{mc^2}$ is in the spectrum of $D^{\mathbb{A}}$ if and only if

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The eigenvalue z has geometric multiplicity equal to $\dim(\text{Ker}(2I + i\mathbb{A}\mathbb{Z}(z)))$ and the corresponding eigenfunction is

$$\psi(x) = \begin{pmatrix} C e^{ik(z)|x|} \\ \tilde{C} \zeta(z)^{-1} \text{sgn}(x) e^{ik(z)|x|} \end{pmatrix}, x \in \mathbb{R} \setminus \{0\},$$

$$(-i(C + \tilde{C}), -i\zeta(z)(C - \tilde{C})) \in \text{Ker}(2I + i\mathbb{A}\mathbb{Z}(z)).$$

Spectral transitions

Theorem

We have

$$\sigma(D^{\mathbb{A}}) = \sigma_p(D^{\mathbb{A}}) \cup \mathbb{R}_{mc^2}$$

and no points from $\sigma_p(D^{\mathbb{A}})$ are in \mathbb{R}_{mc^2} .

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- $m = 0 \wedge \operatorname{tr} \mathbb{A} = 0 \wedge \det \mathbb{A} = 4 \Rightarrow \sigma_p(D^{\mathbb{A}}) = \mathbb{C} \setminus \mathbb{R}.$

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- $m \neq 0 \wedge \det \mathbb{A} = 4 \wedge \alpha = \delta = 0 \Rightarrow \sigma_p(D^{\mathbb{A}}) = \mathbb{C} \setminus \mathbb{R}_{mc^2}.$

In all other cases we have at most two eigenvalues of $D^{\mathbb{A}}.$

Schrödinger operator

$$H_0 = -\frac{1}{2m} \frac{d^2}{dx^2},$$

$$\text{Dom } H_0 = W^{2,2}(\mathbb{R}).$$

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The resolvent

$$K_z(x, y) = \frac{im}{\mu(z)} e^{i\mu(z)|x-y|}.$$

[Grod, Kuzhel 14]

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$H^{\mathbb{A}}$ will be define as

$$H^{\mathbb{A}}\varphi(x) = -\frac{1}{2m} \frac{d^2}{dx^2}\varphi(x), \quad \forall x \in \mathbb{R} \setminus \{0\},$$

$$\text{Dom } H^{\mathbb{A}} = \{\varphi \in W^{2,2}(\mathbb{R} \setminus \{0\}) \mid \Gamma_1\varphi = \mathbb{V}\mathbb{A}\mathbb{V}^*\Gamma_0\varphi\},$$

where

$$\Gamma_0\varphi = \frac{1}{2} \begin{pmatrix} \varphi(0+) + \varphi(0-) \\ -\varphi'(0+) - \varphi'(0-) \end{pmatrix}, \quad \Gamma_1\varphi = \begin{pmatrix} \varphi'(0+) - \varphi'(0-) \\ \varphi(0+) - \varphi(0-) \end{pmatrix},$$

and

$$\mathbb{V} = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}.$$

Non-relativistic limit of point interactions

[Benvegnu, Dabrowski 94] Subtract the rest energy mc^2 and

$$\mathbb{A} \mapsto \mathbb{A}_c = \begin{pmatrix} \frac{1}{2mc}\alpha & \beta \\ \gamma & 2mc\delta \end{pmatrix}.$$

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Theorem

$z \in \mathbb{C} \setminus [0, +\infty)$ such that

$$4 - \det \mathbb{A} + 2i \frac{1}{\mu(z)} \alpha + 2i \mu(z) \delta \neq 0.$$

Then

$$(D^{\mathbb{A}_c} - mc^2 - z)^{-1} \xrightarrow[c \rightarrow +\infty]{u} (H^{\mathbb{A}} - z)^{-1} P_+.$$

$$P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Non-relativistic limit of non-local potentials

Theorem

$$1 + \frac{\tilde{E}_\varepsilon}{\mu(z)}(\alpha + \delta\mu(z)^2) + \tilde{E}_\varepsilon^2 \det \mathbb{A} \neq 0, \quad \tilde{E}_\varepsilon = \frac{i}{2} \int_{\mathbb{R}^2} v_\varepsilon(x) e^{i\mu(z)|x-y|} v_\varepsilon(y) dx dy.$$

$$(D_0 + cV_\varepsilon \mathbb{A}_c - mc^2 - z)^{-1} \xrightarrow[c \rightarrow +\infty]{u} (H_0 + W_\varepsilon^\mathbb{A} - z)^{-1} P_+,$$

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where

$$W_\varepsilon^\mathbb{A} = \frac{1}{2m(1 - \delta\|v_\varepsilon\|_{L^2}^2)} \langle \mathbb{W}_\varepsilon | \hat{\mathbb{A}} \rangle_2 + \frac{1}{2m} \frac{\beta\gamma}{1 - \delta\|v_\varepsilon\|_{L^2}^2} |v_\varepsilon\rangle \langle v_\varepsilon|,$$

where

$$\hat{\mathbb{A}} = \begin{pmatrix} \alpha(1 - \delta\|v_\varepsilon\|_{L^2}^2) & i\beta \\ -i\gamma & \delta \end{pmatrix} \text{ and}$$

$$\mathbb{W}_\varepsilon = \begin{pmatrix} |v_\varepsilon\rangle \langle v_\varepsilon| & |v_\varepsilon\rangle \langle (v_\varepsilon)'| \\ |(v_\varepsilon)' \rangle \langle v_\varepsilon| & |(v_\varepsilon)' \rangle \langle (v_\varepsilon)'| \end{pmatrix}.$$

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Then

$$\begin{array}{ccc} (D_0 + cV_\varepsilon^{\mathbb{A}} - mc^2) & \xrightarrow[u]{\varepsilon \rightarrow 0} & (D^{\mathbb{A}} - mc^2) \\ \downarrow \scriptstyle c \rightarrow +\infty \quad u & & \downarrow \scriptstyle u \quad c \rightarrow +\infty \\ (H_0 + W_\varepsilon^{\mathbb{A}}) & \xrightarrow[\varepsilon \rightarrow 0]{u} & H^{\mathbb{A}} \end{array}$$



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M. Tušek, *Approximation of one-dimensional relativistic point interactions by regular potentials revised*. Letters in Mathematical Physics 110, 2020, 2585-2601.

$$z \in \sigma_p(D_\varepsilon^{\mathbb{A}}) \text{ iff } \det(I + E_\varepsilon \mathbb{A}\mathbb{Z}(z)), E_\varepsilon = \frac{i}{2} \int_{\mathbb{R}^2} v(x) e^{ik(z)\varepsilon|x-y|} v(y) dy dx.$$

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Asymptotic behaviour of eigenvalues of approximations nearby the eigenvalues of the limiting operator

$$\det(I + E_\varepsilon \mathbb{A}\mathbb{Z}(z)) \xrightarrow{\varepsilon \rightarrow 0} \det(I + \frac{i}{2} \mathbb{A}\mathbb{Z}(z)).$$

$$\det(I + E_\varepsilon \mathbb{A}\mathbb{Z}(z)) = \det(I + \frac{i}{2} \mathbb{A}\mathbb{Z}(z)) + O(\varepsilon).$$

$$D_0 = -ic(\sigma_1 \partial_x + \sigma_2 \partial_y) + m\sigma_3,$$

$$\text{Dom } D_0 := H^1(\mathbb{R}^2; \mathbb{C}^2).$$

The smooth boundary $\Sigma := \partial\Omega$. $\mathbb{R}^2 = \Omega_+ \cup \Sigma \cup \Omega_-$. Distribution δ_Σ .

$$D_0 + (\alpha I + \beta \sigma_3 + \gamma(\sigma \cdot \vec{t}) + \delta(\sigma \cdot \vec{n}))\delta_\Sigma.$$

Closed densely defined symmetric operator A on \mathcal{H} . $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ boundary triplet for A^* if and only if

- $\forall f, g \in \text{Dom } A^*$

$$\langle A^* f | g \rangle_{\mathcal{H}} - \langle f | A^* g \rangle_{\mathcal{H}} = \langle \Gamma_1 f | \Gamma_0 g \rangle_{\mathcal{G}} - \langle \Gamma_0 f | \Gamma_1 g \rangle_{\mathcal{G}}.$$

- The map $f \in \text{Dom } A^* \mapsto (\Gamma_0 f, \Gamma_1 f) \in \mathcal{G} \times \mathcal{G}$ is surjective.