Aharonov–Casher theorem on domains with boundary

M. Fialová

talk based on my PhD thesis advised by J.P. Solovej

AAMP September, 2022

Institute of Science and Technology Austria

University of Copenhagen

QMATH

Institute of Science and Technology Austria

Problem: A relativistic massless charged particle in a plane region

Magnetic field $\vec{B} = (0, 0, B)$ (parts B_0; $B_1, B_2, \ldots$)
Vector potential $\vec{a} = (a_x, a_y, 0)$, $(\vec{\nabla} \times \vec{a} = \vec{B}$, $\mathrm{div}\vec{a} = 0)$
Flux $\Phi = \int \vec{B} \cdot \mathrm{d}\vec{S} = \oint \vec{a}\mathrm{d}\vec{s}$

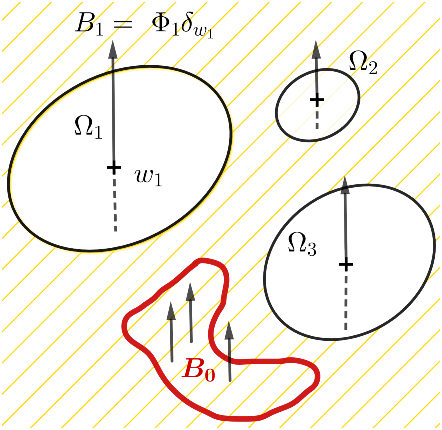


Figure: $\mathbb{R}^2$ with holes

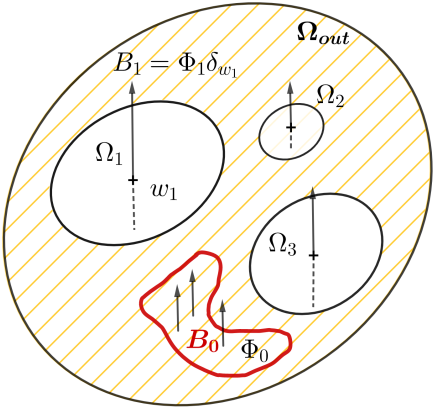


Figure: A disc with holes

(Formal) Hamiltonian for a particle in a plane region

Dirac operator

$$D_a = \sigma^1(-\mathrm{i}\partial_x - a_x) + \sigma^2(-\mathrm{i}\partial_y - a_y)$$

On $\mathbb{C}^2$ valued square integrable functions

Pauli operator

(non-relativistic limit, taking the interaction spin—mag. field into account)

$$D_a^2 = H_a = -\sum_{j\in\{x,y\}} (\partial_j - \mathrm{i}a_j)^2 I + B\sigma^3$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Boundary conditions

Let M be $\mathbb{R}^2$ with holes or a disc with holes and ∂M its boundary
Let n be the coord. in the direction of *inward* normal ∂_n
Rewrite locally: $D_a = \sigma((\partial_n + \mathrm{i}a_n) + A)$, where
$\sigma : \partial M \to \mathrm{End}(\mathbb{C}^2)$,
A is a Dirac operator on ∂M

Denote $D_{max} \subset L^2(M, \mathbb{C}^2)$ the maximal domain of D_a

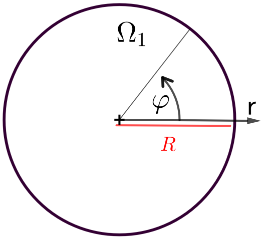
$dom(D_a) = \{u \in dom(D_{max}) \mid u|_{\partial M} \in$ a spectral subspace BC of a boundary operator$\}$

BC is called **boundary condition**

Atiyah–Patodi–Singer (APS) boundary condition

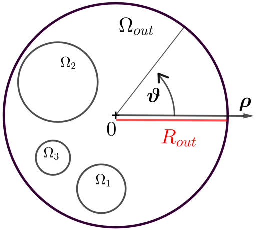
= "the negative spectral subspace of a boundary operator"

APS boundary condition on circular boundaries



$$\begin{pmatrix} u^+ \\ u^- \end{pmatrix}\Bigg|_{\partial\Omega_1} = \left[\sum_{k > \frac{\Phi_1}{2\pi} - \frac{1}{2}} d'_k \begin{pmatrix} e^{i\varphi k} \\ 0 \end{pmatrix} + \sum_{k \leq \frac{\Phi_1}{2\pi} + \frac{1}{2}} b'_k \begin{pmatrix} 0 \\ e^{i\varphi k} \end{pmatrix} \right]$$

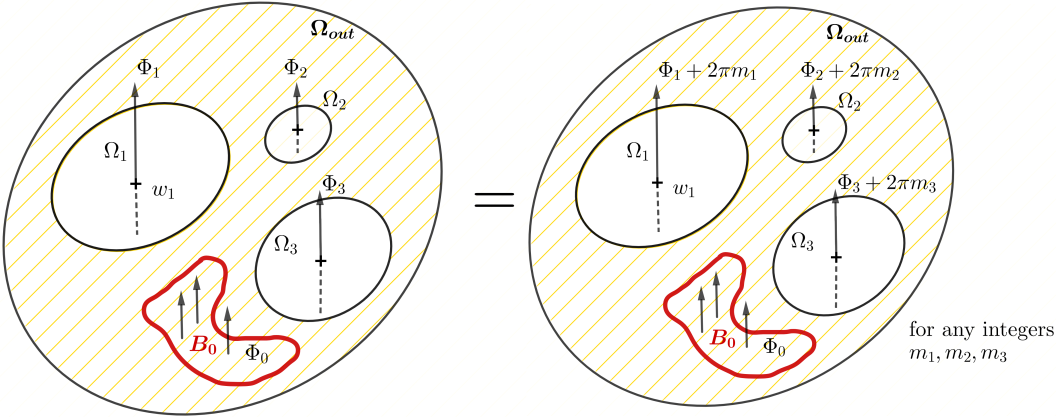
$$\times\ e^{i \int_0^{\varphi} \vec{a}(s)\, d\vec{s} - i \frac{\Phi_1}{2\pi} \varphi}$$



$$\begin{pmatrix} u^+ \\ u^- \end{pmatrix}\Bigg|_{\partial\Omega_{out}} = \left[\sum_{k < \frac{\Phi}{2\pi} - \frac{1}{2}} d''_k \begin{pmatrix} e^{i\vartheta k} \\ 0 \end{pmatrix} + \sum_{k \geq \frac{\Phi}{2\pi} + \frac{1}{2}} b''_k \begin{pmatrix} 0 \\ e^{i\vartheta k} \end{pmatrix} \right]$$

$$\times\ e^{i \int_0^{\vartheta} \vec{a}(s)\, d\vec{s} - i \frac{\Phi}{2\pi} \vartheta}$$

Gauge invariance



We can choose the fluxes inside the holes so that

$$\Phi_{1,2,\ldots,N} \in 2\pi \left[-\tfrac{1}{2}, \tfrac{1}{2}\right).$$

N is the number of the holes

$\lfloor y \rfloor \ldots$ the biggest integer strictly smaller then y,
$\Phi \ldots$ the total flux of the magnetic field a,
ZM$=$ zero modes.

Theorem

Let D_a be the Dirac operator on $\mathbb{R}^2 \setminus \cup_{k \le N} \Omega_k$ with the magnetic field a in the AC gauge. Then if $|\frac{\Phi}{2\pi}| > 1$, the number of ZM of D_a with the APS boundary condition is $\left\lfloor \frac{|\Phi|}{2\pi} \right\rfloor$.

Theorem

Let D_a be the Dirac operator on $\Omega_{out} \setminus \cup_{k \le N} \Omega_k$ with the magnetic field a in the AC gauge. Then the number of ZM of D_a with the APS boundary condition is $\left| \left\lfloor \frac{\Phi}{2\pi} + \frac{1}{2} \right\rfloor \right|$.

If $\Phi > 0 \Rightarrow$ ZM have spin up. If $\Phi < 0 \Rightarrow$ ZM have spin down.

Proof idea [Aharonov–Casher 1979]

$D_a \begin{pmatrix} u^+ \\ u^- \end{pmatrix} = 0$, Ansatz: $u^\pm = \mathrm{e}^{\pm h} g^\pm$. Then

$$0 = \left[\partial_{\bar z} - \frac{\mathrm{i}a}{2}\right] u^+ = \mathrm{e}^{h} \left[\partial_{\bar z} + \underbrace{\partial_{\bar z} h(z) - \frac{\mathrm{i}a}{2}}_{\text{make this 0}}\right] g^+ ,$$

$$0 = \left[\partial_{z} - \frac{\mathrm{i}\bar a}{2}\right] u^- = \mathrm{e}^{-h} \left[\partial_{z} \overbrace{-\partial_{z} h(z) - \frac{\mathrm{i}\bar a}{2}}\right] g^-$$

$\Rightarrow \partial_{\bar z} g^+ = 0 \Rightarrow g^+(z) = \sum_{k\geq 0} d_k z^k$

Aharonov–Casher gauge: $\partial_{\bar z} h(z) = \frac{\mathrm{i}a}{2} \overset{\mathrm{div}\, a = 0}{\overbrace{\Rightarrow}} -\Delta h = B \in C_0^\infty(\mathbb{R}^2)$,

Choose $h(z) = \frac{-1}{2\pi} \int \log|z - z'| B(z')\, \mathrm{d}z'\, \mathrm{d}\bar z'$, $z \notin \mathrm{supp} B$.

$$u^+ = \mathrm{e}^h g^+(z) = |z|^{-\Phi/2\pi} (1 + \mathcal{O}(|z|^{-1})) \sum_{k_0 \geq k \geq 0} d_k z^k \text{, as } |z| \to \infty$$

Extending the Aharonov–Casher idea to the case with boundary

Using the APS boundary condition

- **problem**: function g^+ is analytic only outside of the holes, we have Laurent series $g^+(z) = \sum_{k\in\mathbb{Z}} d_k z^k$
- **solution**: use the boundary condition to find out if d_k vanish for some k
- **means**: multiply u^+ by a convenient function e^G whose restriction to the boundary cancels the exponential term in the boundary condition. The exponential e^{G+h} turns out to be analytic inside Ω_1 and so is, consequently, g^+

$$\mathrm{e}^G u^+ = \mathrm{e}^{G+h} g^+ \qquad \text{by Aharonov–Casher ansatz}$$

$$u^+\big|_{\partial\Omega_1} = \sum_{k>\frac{\Phi_1}{2\pi}-\frac{1}{2}} d_k' \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi k} \\ 0 \end{pmatrix} \times \mathrm{e}^{\mathrm{i}\int_0^\varphi \vec{a}(s)\,\mathrm{d}\vec{s} - \mathrm{i}\frac{\Phi_1}{2\pi}\varphi}, \qquad d_k' \in \mathbb{C}$$

Sphere with N holes

Stereographic projection
Conformal transformation of the Dirac operator with the APS boundary condition
All the fluxes sum to zero

Theorem

Let D_a be the Dirac operator on $\mathbb{S}^2 \setminus \cup_{k \leq N} \Omega_k$ with magnetic field a such that $\int_{\mathbb{S}^2} B = 0$. Denote $\widehat{\Phi} = \Phi'_2 + \cdots \Phi'_N + \Phi_0$ with $[-\pi, \pi) \ni \Phi'_k = \Phi_k + 2\pi m_k$, $k = 2, \ldots, N$ and Φ_0 the flux in the bulk. Then there are

$$\left| \left\lfloor \frac{\widehat{\Phi}}{2\pi} + \frac{1}{2} \right\rfloor \right|$$

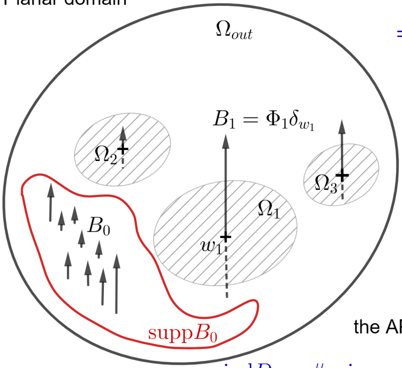
ZM of the operator D_a with the APS boundary conditions.

If $\widehat{\Phi} > 0 \Rightarrow$ ZM have spin up. If $\widehat{\Phi} < 0 \Rightarrow$ ZM have spin down.

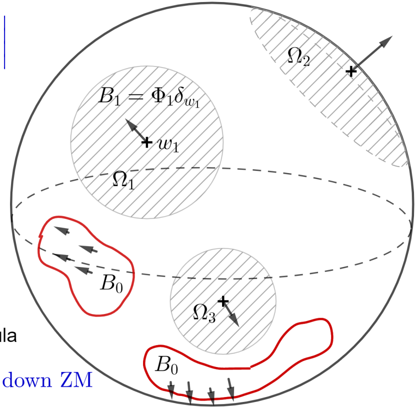
number of zero modes $(\#\mathrm{ZM})$

$$= \left| \left\lfloor \frac{\boldsymbol{\Phi}}{\mathbf{2}\boldsymbol{\pi}} + \frac{\mathbf{1}}{\mathbf{2}} \right\rfloor \right|$$

Planar domain



Sphere with holes



Agrees with

the APS index formula

$$\mathrm{ind} D = \ \#\text{spin up ZM} - \#\text{spin down ZM}$$

Figure: Setting of the bounded region with magnetic field.

Institute of Science and Technology Austria

Concluding remarks

- Dirac operator
- Atiyah–Patodi–Singer boundary condition
- Zero modes (Aharonov–Casher theorem, Index theorem)

- Anomalies in QFT, Domain-Wall-Fermions

Boundary conditions

Let M be a manifold with compact boundary ∂M
Let V be a "suitable" vector space
Let D be a Dirac operator, $D_{max} \subset L^2(M, V)$ its maximal domain
Denote by ∂_n the inward normal vector field on ∂M
Rewrite locally: $D = \sigma(\partial_n + A)$, where

$\sigma : \partial M \to \mathrm{End}(V)$,

A is a Dirac operator on ∂M

$$dom(D) = \{u \in dom(D_{max}) \mid u|_{\partial M} \in \text{a spectral subspace BC of a boundary operator}\}$$

BC is called **boundary condition**

Atiyah–Patodi–Singer (APS) boundary condition

= "the negative spectral subspace of a boundary operator"